On nearly S-paracompactness

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Abstract: - The objective of the present work is to introduce the notion of α-nearly S-paracompact subset, which is closely related to α-nearly paracompact and αS-paracompact subsets. Moreover, we study the invariance under direct and inverse images of open, perfect and regular perfect functions of the nearly S-paracompact spaces [10] and analyze the behavior of such spaces through the sum and topological product.

Key-Words: - semi-open set; regular open set; nearly S-paracompact space; α-nearly S-paracompact subset; perfect function; sum space; product space.

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1 Introduction

The notion of paracompactness was introduced in 1944 by J. Dieudonné [1], succeeding in showing that metrizable locally compact spaces are paracompact. In the present year, several works have been carried out related to modifications of paracompact spaces, which have triggered interesting results, as we can see in [2], [3] and [4].

The notion of α-paracompact subset was introduced in 1966 by C. E. Aull [5] as follows: a subset of a space is said to be α-paracompact if every open cover of the subset has a locally finite open refinement which cover the subset. In 1969, M. K. Singal and S. P. Arya [6] introduced a generalization de paracompactness called nearly paracompact space. This class of spaces was defined using the regular open sets due to M. H. Stone [7] and is characterized by the condition that every regular open cover of the space has a locally finite open refinement which cover the space. Replacing the open cover with a regular open cover in the definition of α-paracompact subset, in 1979, I. Kovăcević [8] introduced the class of α-nearly paracompact subsets, as follows: a subset of a space is said to be α-nearly paracompact if every regular open cover of the subset has a locally finite open refinement which cover the subset. Similarly, in 2006, K. Y. Al-Zoubi [9] introduced the notions of S-paracompact space and αS-paracompact subset. Very recently, J. Sanabria, E. Rosas, and C. Blanco [10] have introduced and investigated the concept of nearly S-paracompact space, which is closely related to nearly paracompact and S-paracompact spaces, in the sense that each nearly paracompact space (resp. S-paracompact) is nearly S-paracompact.

The objective of the present work is to introduce the notion of α-nearly S-paracompact subset, whose definition is similar to the different concepts of subsets mentioned above, but in this case the given cover is formed by regular open sets and the refinement is formed by semi-open sets. Moreover, we study the invariance under direct and inverse images of open, perfect and regular perfect functions of the nearly S-paracompact spaces and analyze the behavior of such spaces through the sum and topological product. This work can be motivating to obtain results and applications in the contexts given in [2], [3] and [4]. Also, it might be of interest to explore new modifications of paracompactness in ideal topological spaces, soft topological spaces and ideal soft topological spaces.

2 Preliminaries

Throughout this paper, (X, τ) always means a topological space (or simply space) on which no separation axioms are assumed unless explicitly stated. If A is a subset of (X, τ), then we denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of (X, τ) is said to be *semi-open* [11] if there exists $U \in \tau$ such that $U \subset A \subset \mathrm{Cl}(U)$. This is equivalent to say that $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. A subset A of (X, τ) is called *regular open* [7] if $A = \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a semi-open (resp. regular open) set is called a *semi-closed* (resp. *regular closed*) set. The collection of all semi-open (resp. regular open) sets of (X, τ) is denoted by

$\mathrm{SO}(X,\tau)$ (resp. $\mathrm{RO}(X,\tau)$). The *semi-closure* [12] of a subset A of (X,τ), denoted by $\mathrm{sCl}(A)$, is defined by the intersection of all semi-closed sets containing A, and the *δ-closure* of A [13], denoted by $\mathrm{Cl}_\delta(A)$, is defined as the set of all points $x \in X$ such that $A \cap \mathrm{Int}(\mathrm{Cl}(U)) \neq \emptyset$ for each open set U containing x. A subset A of (X,τ) is said to be *δg-closed* [14] (resp. *sg-closed* [15]) if $\mathrm{Cl}_\delta(A) \subset U$ (resp. $\mathrm{sCl}(A) \subset U$) whenever $A \subset U$ y $U \in \tau$ (resp. $U \in \mathrm{SO}(X,\tau)$). Also, a subset A of (X,τ) is called *θs-open* [9] if for each $x \in A$ there exists $U \in \tau$ such that $x \in U \subset \mathrm{sCl}(U) \subset A$. The complement of a θs-open set is called a *θs-closed* set. A space (X,τ) is said to be *nearly S-paracompact* [10], if every regular open cover of X has a locally finite semi-open refinement which covers to X (we do not require a refinement to be a cover). A space (X,τ) is said to be *nearly compact* [16] if every regular open cover of X has a finite subcover. It is well known that, if Y is a subset of (X,τ), the relative topology induced on Y by τ, denoted by τ_Y, is the collection $\tau_Y = \{U \cap Y : U \in \tau\}$. The following lemmas will be useful in the sequel.

Lemma 2.1. [17, Lemma 4] *Let (X,τ) be a space. If Y is an open or dense subset of X, then:*

1. $\mathrm{RO}(Y,\tau_Y) = \{V \cap Y : V \in \mathrm{RO}(X,\tau)\}$.
2. $(\tau_Y)_s = (\tau_s)_Y$.

The proof of the next lemma immediately follow from Lemma 2.1.

Lemma 2.2. *Let (X,τ) be a space and $U \subset Y \subset X$, where $Y \in \mathrm{RO}(X,\tau)$. If $U \in \mathrm{RO}(Y,\tau_Y)$, then $U \in \mathrm{RO}(X,\tau)$.*

The proofs of the next two lemmas immediately follow from [12, Theorem 2.13] and [18, Theorem 1.9].

Lemma 2.3. *Let (X,τ) be a space. If $Y \in \tau$ and $U \in \mathrm{SO}(X,\tau)$, then $Y \cap U \in \mathrm{SO}(Y,\tau_Y)$.*

Lemma 2.4. *Let (X,τ) be a space and $A \subset Y \subset X$, where $Y \in \tau$. If $A \in \mathrm{SO}(Y,\tau_Y)$, then $A \in \mathrm{SO}(X,\tau)$.*

3 Properties of α-nearly S-paracompact subsets

According to [9], a subset A of (X,τ) is called αS-paracompact if every open cover of A has a locally finite semi-open refinement which covers A. In this section, we introduce the notion of α-nearly S-paracompact subset, which is closely related to α-nearly paracompact and αS-paracompact subsets.

Definition 3.1. A subset A of a space (X,τ) is said to be *α-nearly S-paracompact*, if every cover $\mathcal{U}$ of A by regular open subsets of X has a locally semi-open refinement $\mathcal{V}$ which covers A.

Obviously, every αS-paracompact subset is α-nearly S-paracompact and every α-nearly paracompact subset is α-nearly S-paracompact, but the converse is not necessarily true as we will see in Examples 3.2 and 3.3, below.

Example 3.2. There exist a space X and an α-nearly S-paracompact subset A of X that is not αS-paracompact in X. Let $X = \{a, b, c, a_i : i = 1, 2, \ldots\}$. Topologize X as follows: the points c and a_i are isolated; the fundamental system of neighborhoods of a is $\{O_n(a) : n = 1, 2, \ldots\}$, where $O_n(a) = \{a, a_i : i \geq n\}$; the fundamental system of neighborhoods of b is $\{O_n(b) : n = 1, 2, \ldots\}$, where $O_n(b) = O_n(a) \cup \{b, c\}$. I. Kovăcević [19] showed that the set $A = \{b, a_i : i = 1, 2, \ldots\}$ is α-nearly paracompact in X and, hence, it is α-nearly S-paracompact. On the other hand, A is not αS-paracompact in X, because if we consider the cover $\mathcal{V} = \{O_n(b) : n = 1, 2, \ldots\}$ of A by open subsets of X, then every semi-open refinement of $\mathcal{V}$ which covers A is not locally finite at the point b, it follows that A is not αS-paracompact in X. ■

Example 3.3. There exist a space X and an α-nearly S-paracompact subset A of X that is not α-nearly paracompact in X. Let $X = \mathbb{R}^+ \cup \{p\}$, where $\mathbb{R}^+ = [0, +\infty)$ and $p \notin \mathbb{R}^+$. Topologize X as follows: $\mathbb{R}^+$ has the usual topology and is an open subspace of X; a basic neighborhood of $p \notin X$ takes the form $O_n(p) = \{p\} \cup \bigcup_{i=n}^{\infty} (2i, 2i+1)$, where $n \in \mathbb{N}$. P.-Y. Li and Y.-K. Song [20] showed that $A = \{p\} \cup \bigcup_{n=0}^{\infty} [2n, 2n+1]$ is an αS-paracompact subset of X and hence, it is α-nearly S-paracompact in X. On the other hand, A is not α-nearly paracompact in X, because we consider the cover $\mathcal{V} = \left\{ \left[0, \frac{1}{3}\right) \right\} \cup \bigcup_{i=1}^{\infty} \left\{ \left(i - \frac{1}{3}, i + \frac{1}{3}\right) \right\} \cup \left\{ \bigcup_{i=0}^{\infty} (2i, 2i+1) \cup \{p\} \right\}$ of A by regular open subsets of X, then every open refinement of $\mathcal{V}$ which covers A is not locally finite at the point p, it follows that A is not α-nearly paracompact in X. ■

Theorem 3.4. *If A is a δg-closed subset of a nearly S-paracompact space (X,τ), then A is α-nearly S-paracompact in (X,τ).*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a cover of A by regular open subsets of X. Because $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$ and

A is δg-closed, we have $\mathrm{Cl}_\delta(A) \subset \bigcup_{\lambda\in\Lambda} U_\lambda$. For each $x \notin \mathrm{Cl}_\delta(A)$, there exists an open set W_x such that $x \in W_x$ and $A \cap \mathrm{Int}(\mathrm{Cl}(W_x)) = \emptyset$. Let $\mathcal{U}' = \{U_\lambda : \lambda \in \Lambda\} \cup \{\mathrm{Int}(\mathrm{Cl}(W_x)) : x \notin \mathrm{Cl}_\delta(A)\}$. Then, $\mathcal{U}'$ is a regular open cover of the nearly S-paracompact space (X,τ), it follows that $\mathcal{U}'$ has a locally finite semi-open refinement $\mathcal{V} = \{V_\beta : \beta \in \Delta\}$ which covers X. For each $\beta \in \Delta$, we have $V_\beta \subset U_{\lambda(\beta)}$ for some $\lambda(\beta) \in \Lambda$ or $V_\beta \subset \mathrm{Int}(\mathrm{Cl}(W_{x(\beta)}))$ for some $x(\beta) \notin \mathrm{Cl}_\delta(A)$. Then, the colection $\mathcal{V}' = \{V_\beta : \beta \in \Delta_0\}$ where $\Delta_0 = \{\beta \in \Delta : V_\beta \subseteq U_{\lambda(\beta)}\}$ is a locally finite semi-open refinement of $\mathcal{U}$. We assert $A \subset \bigcup_{\beta\in\Delta_0} V_\beta$. If $z \notin \bigcup_{\beta\in\Delta_0} V_\beta$, then $z \notin V_\beta$ for each $\beta \in \Delta_0$ and, as $\mathcal{V}$ is a cover of X, there exists $\beta_0 \notin \Delta$ such that $z \in V_{\beta_0} \subset \mathrm{Int}(\mathrm{Cl}(W_{x(\beta_0)}))$, with $x(\beta_0) \notin \mathrm{Cl}_\delta(A)$. Since $A \cap \mathrm{Int}(\mathrm{Cl}(W_{x(\beta_0)})) = \emptyset$, we conclude that $z \notin A$ and consequently, $A \subset \bigcup_{\beta\in\Delta_0} V_\beta$. This shows that A is an α-nearly S-paracompact subset of (X,τ). □

Theorem 3.5. *Let (X,τ) be a space. Then, the following statements hold:*

1. *If A is an open α-nearly S-paracompact subset of (X,τ), then (A,τ_A) is nearly S-paracompact.*
2. *If A is a clopen subset of (X,τ), then A is α-nearly S-paracompact in (X,τ) if and only if (A,τ_A) is nearly S-paracompact.*

Proof. (1) Assume that A is an open α-nearly S-paracompact subset of a space (X,τ). Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a cover of A by regular open sets in (A,τ_A). Since A is open in (X,τ), by Lemma 2.1, it follows that $U_\lambda = V_\lambda \cap A$, with $V_\lambda \in \mathrm{RO}(X,\tau)$, for each $\lambda \in \Lambda$. Clearly, $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a cover of A by regular open sets in (X,τ) and, as A is α-nearly S-paracompact, there exists a collection $\mathcal{W} = \{W_\beta : \beta \in \Delta\}$ of semi-open sets in (X,τ) such that $\mathcal{W}$ is locallly finite in (X,τ) and is a refinement of $\mathcal{V}$ which covers A. It is easy to see that $\mathcal{W}_A = \{W_\beta \cap A : \beta \in \Delta\}$ is a locally finite collection of semi-open sets in (A,τ_A) such that $\mathcal{W}_A$ is a refinement of $\mathcal{U}$ which covers A. Therefore, (A,τ_A) is a nearly S-paracompact space.

(2) If A is a clopen α-nearly S-paracompact subset of (X,τ), then by (1), we obtain (A,τ_A) is a nearly S-paracompact space. Conversely, let $\mathcal{U} = \{U_\beta : \beta \in \Delta\}$ be a cover of A by regular open sets in (X,τ). By Lemma 2.1, $\mathcal{U}_A = \{U_\beta \cap A : \beta \in \Delta\}$ is a collection of regular open sets in (A,τ_A) and, as (A,τ_A) is a nearly S-paracompact space, there exists a collection $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of semi-open sets in (A,τ_A) such that $\mathcal{W}$ is locally finite in (A,τ_A) and is a refinement of $\mathcal{U}$ which covers A. The remainder of proof is similar to that of [9, Theorem 3.4]. □

Corollary 3.6. *Every clopen subspace of a nearly S-paracompact space is nearly S-paracompact.*

Proof. Since every clopen set is δg-closed, the proof follows from Theorems 3.4 and 3.5. □

Theorem 3.7. *If (X,τ) is a Hausdorff space and A is an α-nearly S-paracompact subset of (X,τ), then A is θs-closed.*

Proof. Let $x \notin A$. Because (X,τ) is Hausdorff, for each $y \in A$ there exists an open set U_y containing y such that $x \notin \mathrm{Cl}(U_y)$. Since $U_y \subset \mathrm{Int}(\mathrm{Cl}(U_y))$, it follows that $\mathcal{U} = \{\mathrm{Int}(\mathrm{Cl}(U_y)) : y \in A\}$ is a cover of A by regular open subsets of X and, as A is α-nearly S-paracompact, there exists a collection $\mathcal{V}$ of semi-open sets in X such that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$ which covers A. The remainder of proof is similar to that of [9, Theorem 3.7]. □

Theorem 3.8. *Let (X,τ) be a regular space and A be a subset of X. Then, A is αS-paracompact in (X,τ) if and only if A is α-nearly S-paracompact in (X,τ).*

Proof. If A is an αS-paracompact subset of (X,τ), the A is α-nearly S-paracompact in (X,τ), because every regular open set is open. For the converse, let $\mathcal{U} = \{U_\beta : \beta \in \Delta\}$ a cover of A by open subsets of (X,τ). For each $x \in A$, there exists a $\beta(x) \in \Delta$ such that $x \in U_{\beta(x)}$ and, as (X,τ) is a regular space, there exists an open subset V_x of (X,τ) such that $x \in V_x \subset \mathrm{Cl}(V_x) \subset U_{\beta(x)}$. Thus, $\mathcal{V} = \{V_x : x \in A\}$ is a cover of A by open subsets of (X,τ) and, as $V_x \subset \mathrm{Int}(\mathrm{Cl}(V_x))$, we have $\mathcal{V}' = \{\mathrm{Int}(\mathrm{Cl}(V_x)) : x \in A\}$ is a cover of A by regular open sets in (X,τ). Since A is α-nearly S-paracompact, there exists a collection $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of semi-open sets in (X,τ) such that $\mathcal{W}$ is a locally finite refinement of $\mathcal{V}'$ which covers A. Also, for each $\lambda \in \Lambda$, there exists a $x(\lambda) \in A$ such that $W_\lambda \subset \mathrm{Int}(\mathrm{Cl}(V_{x(\lambda)})) \subset \mathrm{Int}(U_{\beta(x(\lambda))}) = U_{\beta(x(\lambda))}$, it follows that $\mathcal{W}$ is a refinement of $\mathcal{U}$. Hence, A is αS-paracompact in (X,τ). □

It is known that in the presence of the axiom of regularity, the notions of α-paracompact and αS-paracompact subsets coincide (see [20, Theorem 2.13]). According to the previous result, we have the following corollary.

Corollary 3.9. *Let (X,τ) be a regular space and A be a subset of X. Then, the following statements are equivalent:*

1. *A is α-paracompact.*

2. *A is αS-paracompact.*

3. *A is α-nearly S-paracompact.*

The proof of the following result can be obtained as that of [9, Proposition 3.9].

Proposition 3.10. *Let A and B be two subsets of (X, τ) such that $A \subset B \subset \mathrm{sCl}(A)$. If A is sg-closed and α-nearly S-paracompact, then B is α-nearly S-paracompact.*

Proposition 3.11. *Let A and B be two subsets of (X, τ) such that $A \subset B$ and B is clopen. Then A is α-nearly S-paracompact in (B, τ_B) if and only if A is α-nearly S-paracompact in (X, τ).*

Proof. Suppose that A is α-nearly S-paracompact in (B, τ_B). If $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ is a cover of A by regular open sets in (X, τ), then $A \subset B \cap \left(\bigcup_{\lambda \in \Lambda} U_\lambda \right) = \bigcup_{\lambda \in \Lambda} (B \cap U_\lambda)$. By Lemma 2.1, $\mathcal{U}' = \{B \cap U_\lambda : \lambda \in \Lambda\}$ is a cover of A by regular open sets in (B, τ_B) and, hence, there exists a collection $\mathcal{V} = \{V_\beta : \beta \in \Delta\}$ of semi-open sets in (B, τ_B) such that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}'$ which covers A. Using Lemma 2.4, we obtain $\mathcal{V}$ is a collection of semi-open sets in (X, τ) such that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$. Therefore A is α-nearly S-paracompact in (X, τ). Conversely, assume that A is α-nearly S-paracompact in (X, τ) and let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a cover of A by regular open sets in (B, τ_B). Since $B \in \mathrm{RO}(X, \tau)$, by Lemma 2.2, it follows that $\mathcal{U}$ is a cover of A by regular open sets in (X, τ) and, hence, there exists a collection $\mathcal{V} = \{V_\beta : \beta \in \Delta\}$ of semi-open sets in (X, τ) such that $\mathcal{V}$ is locally finite in (X, τ) and is a refinement of $\mathcal{U}$ which covers A. Because $B \in \tau$, by Lemma 2.3, we obtain $\mathcal{V}_B = \{V_\beta \cap B : \beta \in \Delta\}$ is a collection of semi-open sets in (B, τ_B). It can easily be shown that $\mathcal{V}_B = \{V_\beta \cap B : \beta \in \Delta\}$ is locally finite in (B, τ_B) and is a refinement of $\mathcal{U}$ which covers A. Hence A is α-nearly S-paracompact in (B, τ_B). □

4 Invariance under perfect functions

In this section, we study the invariance of nearly S-paracompact spaces under direct and inverse images of open, perfect, and regular perfect functions. In what follows, we consider that X and Y are sets that are endowed with respective topologies. Furthermore, for a function $f : X \to Y$ and a point $y \in Y$, the inverse image $f^{-1}(\{y\})$ is denoted by $f^{-1}(y)$. Recall that a function $f : X \to Y$ is called *perfect* [21] if f is surjective, continuous, closed and the fiber $f^{-1}(y)$ is compact for each $y \in Y$. A function $f : X \to Y$ is called *almost continuous* [22] (resp. almost completely continuous [23]) if $f^{-1}(U)$ is an open (resp. regular open) subset of X for every regular open subset U of Y. Also, $f : X \to Y$ is called *almost open* [22] if $f(U)$ is an open subset of Y for every regular open subset U of X. Next, we introduce the classes of regular closed, regular open and regular perfect functions.

Definition 4.1. *A function $f : X \to Y$ is said to be* regular closed *(resp.* regular open*), if $f(U)$ is a regular closed (resp. regular open) subset of Y for every closed (resp. open) subset U of X.*

We says that a function $f : X \to Y$ is *regular perfect*, if f is surjective, continuous, regular closed and the fiber $f^{-1}(y)$ is compact for each $y \in Y$.

Remark 4.2. *From the definitions we have the following facts:*

1. *A function $f : X \to Y$ is almost continuous (resp. almost completely continuous) if and only if $f^{-1}(B)$ is a closed (resp. regular closed) subset of X for every regular closed subset B of Y.*

2. *Every open function is almost open and, every continuous function is almost continuous.*

3. *Every regular closed function is closed. Also, every regular open function is open and, hence, it is almost open.*

The proof of the following result is easy and therefore omitted.

Lemma 4.3. *For a function $f : X \to Y$ consider the following statemens:*

1. *f is regular closed.*

2. *For each set $B \subset Y$ and each open set $U \subset X$ containing $f^{-1}(B)$, there exists a regular open set $V \subset Y$ such that $B \subset V$ and $f^{-1}(V) \subset U$.*

3. *For each point $y \in Y$ and each open set $U \subset X$ containing $f^{-1}(y)$, there exists a regular open set $V \subset Y$ such that $y \in V$ and $f^{-1}(V) \subset U$.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

The following example shows that the converse of implication $(1) \Rightarrow (3)$ in Lemma 4.3, in general, is not true.

Example 4.4. There exists a function that satisfies part (3) of the previous Lemma but that is not regular closed. Let $\mathbb{R}$ be the set of real numbers and consider τ_u and τ_c the usual topology and the cofinite topology on $\mathbb{R}$, respectively. Let $f : (\mathbb{R}, \tau_c) \to (\mathbb{R}, \tau_u)$ be the identity function. It is easy to see that part (3) of the previous lemma is satisfied, because

$\tau_c \subset \tau_u$ and the basics in τ_u are regular open sets in τ_u. On the other hand, as $\{0\}$ is a closed set in $(\mathbb{R}, \tau_c)$ and $f(\{0\})$ is not a regular closed set in $(\mathbb{R}, \tau_u)$, we have f is not a regular closed function. ■

To achieve the objectives of this section we will use the following three Lemmas.

Lemma 4.5. [22, Lemma 1] *If $f : X \to Y$ is an almost continuous and almost open function, then it is almost completely continuous.*

Lemma 4.6. [24, Lemma 3.1] *Let $f : X \to Y$ an open and continuous function. If U is a semi-open subset of Y and V is an open subset of X, then $f^{-1}(U) \cap V$ is a semi-open subset of X.*

Lemma 4.7. [25, Lemma 3.10.11] *If $f : X \to Y$ is a perfect function and $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a locally finite collection of subsets of X, then $f(V) = \{f(V_\lambda) : \lambda \in \Lambda\}$ is a locally finite collection of subsets of Y.*

Theorem 4.8. *Let $f : X \to Y$ be an open and perfect function. If X is nearly S-paracompact, then Y is nearly S-paracompact.*

Proof. Asume that X is a nearly S-paracompact space and let $\mathcal{U} = \{U_\beta : \beta \in \Delta\}$ be a regular open cover of Y. Since the hypotheses of Lemma 4.5 are satisfied, the function f is almost completely continuous and so, $f^{-1}(\mathcal{U}) = \{f^{-1}(U_\beta) : \beta \in \Delta\}$ is a regular open cover of X. Hence, $f^{-1}(\mathcal{U})$ has a locally finite semi-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ which covers X. For each $\lambda \in \Lambda$ there exists an open set O_λ such that $O_\lambda \subset V_\lambda \subset \mathrm{Cl}(O_\lambda)$ and by continuity of f we have $f(O_\lambda) \subset f(V_\lambda) \subset f(\mathrm{Cl}(O_\lambda)) \subset \mathrm{Cl}(f(O_\lambda))$. Since f is open, it follows that $f(O_\lambda)$ is an open set for each $\lambda \in \Lambda$ and, so, $f(V_\lambda)$ is a semi-open set in Y for each $\lambda \in \Lambda$. Therefore, $f(\mathcal{V}) = \{f(V_\lambda) : \lambda \in \Lambda\}$ is a semi-open refinement of $\mathcal{U}$ which covers Y. Finally, By Lema 4.7, we conclude that $f(\mathcal{V})$ is locally finite in Y. This shows that Y is a nearly S-paracompact space. □

Theorem 4.9. *Let $f : X \to Y$ be an open and regular perfect function. If Y is nearly S-paracompact, then X is nearly S-paracompact.*

Proof. Suppose that Y is a nearly S-paracompact space and let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be regular open cover of X. For each $y \in Y$, we have $f^{-1}(y) \subset X \subset \bigcup_{\lambda\in\Lambda} U_\lambda$, which tells us that $\mathcal{U}$ is an open cover of $f^{-1}(y)$ for each $y \in Y$. The compactness of $f^{-1}(y)$ guarantees the existence of a finite subcollection $\mathcal{U}_y = \{U_{\lambda_1}(y), U_{\lambda_2}(y), \cdots, U_{\lambda_n}(y)\}$ of $\mathcal{U}$ which covers $f^{-1}(y)$. Since $O_y = \bigcup_{i=1}^{n} U_{\lambda_i}(y)$ is an open set such that $f^{-1}(y) \subset O_y$ and, as f is a regular closed function, then by Lemma 4.3, there exists a regular open set V_y in Y such that $y \in V_y$ and $f^{-1}(V_y) \subset O_y$. Thus, $\mathcal{V} = \{V_y : y \in Y\}$ is a regular open cover of Y and, hence it has a locally finite semi-open refinement which covers Y. By [26, Lemma 1.3] (case $\mathcal{I} = \{\emptyset\}$), there exists a locally finite semi-open cover $\mathcal{W} = \{W_y : y \in Y\}$ of Y such that $W_y \subset V_y$. Put $\mathcal{G}_y = \{U_{\lambda_i}(y) \cap f^{-1}(W_y) : i = 1, 2, 3..., n\}$. Using the Lemma 4.6 and proceeding as in the proof of [24, Theorem 3.2], we conclude that $\mathcal{G} = \bigcup_{y\in Y} \mathcal{G}_y$ is a locally finite semi-open refinement of $\mathcal{U}$ which covers X. Therefore, X is a nearly S-paracompact space. □

5 Stability under sum and product

In this section, we study the behavior of the sum and the product of spaces when one of the factors is a nearly S-paracompact space.

Theorem 5.1. *The sum $\bigoplus_{\lambda\in\Lambda} X_\lambda$ is nearly S-paracompact if and only if X_λ is nearly S-paracompact, for each $\lambda \in \Lambda$.*

Proof. Assume that $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ is nearly S-paracompact. Since every X_λ is clopen in X, by Corollary 3.6, we conclude that for each $\lambda \in \Lambda$, X_λ is a nearly S-paracompact space. Conversely, assume that X_λ is a nearly S-paracompact space for each $\lambda \in \Lambda$ and let $\mathcal{U}$ be a regular open cover of $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$. By Lemma 2.1, we have for each $\lambda \in \Lambda$, $\mathcal{U}_\lambda = \{U \cap X_\lambda : U \in \mathcal{U}\}$ is a regular open cover of X_λ. Hence, $\mathcal{U}_\lambda$ has a locally finite semi-open refinement $\mathcal{V}_\lambda$ which covers X_λ. Let us define $\mathcal{V} = \bigcup_{\lambda\in\Lambda} \mathcal{V}_\lambda$. Note that $\mathcal{V}$ is a collection of semi-open sets in X such that $X = \bigcup_{\lambda\in\Lambda} X_\lambda \subset \bigcup_{\lambda\in\Lambda} \bigcup_{V\in\mathcal{V}_\lambda} V = \bigcup_{V\in\mathcal{V}} V$, that is, $\mathcal{V}$ is a semi-open cover of X. On the other hand, if $V \in \mathcal{V}$, then there exists a $\lambda \in \Lambda$ such that $V \in \mathcal{V}_\lambda$, and so, $V \subset U \cap X_\lambda \subset U$ for some $U \in \mathcal{U}$, which tells us that $\mathcal{V}$ is a refinement of $\mathcal{U}$. Finally we will show that $\mathcal{V}$ is locally finite in $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$. Let $x \in X$. Then $x \in X_{\lambda_0}$ for some $\lambda_0 \in \Lambda$, and as $\mathcal{V}_{\lambda_0}$ is locally finite in en X_{λ_0}, there exists an open set B in X_{λ_0} such that $x \in B$ and $\{V \in \mathcal{V}_{\lambda_0} : V \cap B \neq \emptyset\}$ is a finite set. Now, $(B \cap X_{\lambda_0}) \cap X_\lambda = B \cap X_{\lambda_0}$ if $\lambda = \lambda_0$, and $(B \cap X_{\lambda_0}) \cap X_\lambda = \emptyset$ if $\lambda \neq \lambda_0$. Then $(B \cap X_{\lambda_0}) \cap X_\lambda$ is an open set in X_λ for each

$\lambda \in \Lambda$, and so, $B = B \cap X_{\lambda_0}$ is an open set in $\bigoplus_{\lambda\in\Lambda} X_\lambda$ such that $x \in B$. Also, if $W \in \mathcal{V}$ and $W \in \mathcal{V}_{\lambda_0}$, then $\{W \in \mathcal{V} : W \cap B \neq \emptyset\}$ is a finite set. Otherwise, if $W \in \mathcal{V}$ and $W \notin \mathcal{V}_{\lambda_0}$, then $W \in \mathcal{V}_\lambda$ con $\lambda \neq \lambda_0$ and so $W \cap B \subset X_\lambda \cap X_{\lambda_0} = \emptyset$, it follows that $\{W \in \mathcal{V} : W \cap B \neq \emptyset\} = \emptyset$ is a finite set. This shows that the collection $\mathcal{V}$ is locally finite in $\bigoplus_{\lambda\in\Lambda} X_\lambda$ and hence, we conclude that $\bigoplus_{\lambda\in\Lambda} X_\lambda$ is a nearly S-paracompact space. □

Next, we consider the Cartesian product $X \times Y$ of two spaces X and Y, endowed with the product topology. The following two lemmas will be useful to study the behavior of the Cartesian product $X \times Y$ when one of the factors is a space nearly S-paracompact. The proofs are easy and therefore omitted.

Lemma 5.2. *For every pair $(x, y) \in X \times Y$ and every regular open subset U of $X \times Y$ containing (x, y), there exist two regular open sets W and V in X and Y, respectively, such that $(x, y) \in W \times V \subset U$.*

Lemma 5.3. *If W is a semi-open subset of X and V is a semi-open subset of Y, then $W \times V$ is a semi-open subset of the product space $X \times Y$.*

Theorem 5.4. *If X is a nearly S-paracompact space and Y is a nearly compact space, then the product space $X \times Y$ is nearly S-paracompact.*

Proof. Let $\mathcal{U}$ be a regular open cover of $X \times Y$. For each pair $(x, y) \in X \times Y$, there exists $U \in \mathcal{U}$ such that $(x, y) \in U$, so by Lemma 5.2, there exist two regular open sets $W_{(x,y)}$ and $V_{(x,y)}$ in X and Y, respectively, such that $(x, y) \in W_{(x,y)} \times V_{(x,y)} \subset U$. Let $I_x = \{x\} \times Y$ for each $x \in X$. Then $\{V_{(x,y)} : (x, y) \in I_x\}$ is a regular open cover of Y, and as Y is a nearly compact space, there exists a finite subset J_x of I_x, such that $\mathcal{V} = \{V_{(x,y)} : (x, y) \in J_x\}$ is a cover of Y. On the other hand, for each $x \in X$, put $M_x = \bigcap_{(x,y)\in J_x} W_{(x,y)}$. Then, M_x is a regular open set containing x and the collection $\mathcal{W} = \{M_x : x \in X\}$ is a regular open cover of X. Since X is a nearly S-paracompact space, there exists a locally finite semi-open refinement $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ of $\mathcal{W}$ which covers X. Now, put $\mathcal{H} = \{G_\lambda \times V_{(x,y)} : \lambda \in \Lambda, (x, y) \in J_x\}$. We will show that $\mathcal{H}$ is a locally finite semi-open refinement of $\mathcal{U}$ which covers $X \times Y$. Indeed:

- For each $\lambda \in \Lambda$ and each $(x, y) \in J_x$, there exists a set $M_x \in \mathcal{W}$ such that $G_\lambda \subset M_x \subset W_{(x,y)}$, which implies that $G_\lambda \times V_{(x,y)} \subset W_{(x,y)} \times V_{(x,y)} \subset U$ for some $U \in \mathcal{U}$, so $\mathcal{H}$ is a refinement of $\mathcal{U}$.
- If $(x, y) \in X \times Y$, then $x \in G_\lambda$ for some $\lambda \in \Lambda$ and $y \in V_{(x,y)}$ for some $(x, y) \in J_x$. Thus, $(x, y) \in G_\lambda \times V_{(x,y)}$, and hence, $\mathcal{H}$ is a cover of $X \times Y$.
- Since $V_{(x,y)}$ is a regular open set in Y, then it is semi-open, and by Lemma 5.3, we have $G_\lambda \times V_{(x,y)}$ is a semi-open set in $X \times Y$, for each $\lambda \in \Lambda$ and each $(x, y) \in J_x$. Therefore, $\mathcal{H}$ is a collection of semi-open sets in $X \times Y$.
- If $(x, y) \in X \times Y$, then there exists an open subset O_x of X such that $x \in O_x$ and O_x intersects a finite number of elements of $\mathcal{G}$. Also, there exists an open subset D_y of Y such that $y \in D_y$ and D_y intersects a finite number of elements of $\mathcal{V}$. Thus, $O_x \times D_y$ is an open subset of $X \times Y$ such that $(x, y) \in O_x \times D_y$ and $O_x \times D_y$ intersects a finite number of elements of $\mathcal{H}$. Therefore, $\mathcal{H}$ is locally finite.

This shows that $\mathcal{H}$ is a locally finite semi-open refinement of $\mathcal{U}$ which covers $X \times Y$ and hence, $X \times Y$ is nearly S-paracompact. □

6 Conclusion

Paracompactness is one of the most useful notions in general topology, because many of the spaces studied in this branch of mathematics are paracompacts. This notion and its various generalizations (nearly paracompactnes, almost paracompactness, S-paracompactnes, etc.) describe the relation between a locally finite property and an entire property of spaces. In this work, we have used the classes of regular open sets and semi-open sets to introduce and study new generalizations of paracompactness of a subset and a space, called α-nearly paracompact subset and nearly S-paracompact space. We have established the most relevant properties of the α-nearly paracompact subsets and their relation to the α-nearly paracompact and αS-paracompact subsets. We show that nearly S-paracompactness is invariant under the direct image of a perfect open function, and is also invariant under the inverse image of a perfect regular open function. We analyze the behavior of nearly S-paracompact spaces through the sum of topological spaces and show that the Cartesian product of a nearly S-paracompact space by a nearly compact space is a nearly S-paracompact space. Regarding this last result, we must point out that it was not possible to use Theorem 4.9 to analyze the behavior of the product $X \times Y$ of a nearly S-paracompact space X and a nearly compact space Y, because the projection $\pi_2 : X \times Y \to Y$ is not necessarily a perfect regular function (in fact it is not a closed regular). Due to the fact that various modifications of the notion of paracompactness have been approached in the contexts of ideal topological spaces ([26], [27]), soft topological spaces ([28], [29]) and soft ideal topological spaces [30], we consider that the notions and results given in this article can be studied in the aforementioned contexts, as follows:

1. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be α-nearly S-paracompact modulo $\mathcal{I}$ if for any regular open cover $\mathcal{U}$ of A, there exist $I \in \mathcal{I}$ and a locally finite collection $\mathcal{V}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V : V \in \mathcal{V}\} \cup I$. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be nearly S-paracompact with respect to $\mathcal{I}$ if X is α-nearly S-paracompact modulo $\mathcal{I}$.
2. Let (X, τ, E) be a soft topological space. A soft set A over X is said to be soft α-nearly S-paracompact if for any soft regular open cover $\mathcal{U}$ of A, there exists a soft locally finite family $\mathcal{V}$ of soft semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V : V \in \mathcal{V}\}$. A sof topological space (X, τ, E) is said to be soft nearly S-paracompact if $\tilde{X}$ is soft α-nearly S-paracompact.
3. Let (X, τ, E) be a soft topological space and let $\mathcal{I}$ be a soft ideal on X. A soft set A over X is said to be soft α-nearly S-paracompact modulo $\mathcal{I}$ if for any soft regular open cover $\mathcal{U}$ of A, there exist $I \in \mathcal{I}$ and a soft locally finite collection $\mathcal{V}$ of soft semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V : V \in \mathcal{V}\} \cup I$. A soft ideal topological space $(X, \tau, E, \mathcal{I})$ is said to be soft nearly S-paracompact with respect to $\mathcal{I}$ if $\tilde{X}$ is soft α-nearly S-paracompact modulo $\mathcal{I}$.

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