A NOTE ON PRESERVATION OF SPECTRA FOR TWO GIVEN OPERATORS

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Abstract. We study the relationships between the spectra derived from Fredholm theory corresponding to two given bounded linear operators acting on the same space. The main goal of this paper is to obtain sufficient conditions for which the spectra derived from Fredholm theory and other parts of the spectra corresponding to two given operators are preserved. As an application of our results, we give conditions for which the above mentioned spectra corresponding to two multiplication operators acting on the space of functions of bounded p-variation in Wiener's sense coincide. Additional illustrative results are given too.

Keywords: restriction of an operator; spectral property; semi-Fredholm spectra; multiplication operator

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1. Introduction

In [5], Barnes studied the relationship between the spectral and Fredholm properties of an operator and the Fredholm properties of its extensions to certain superspaces, assuming some special conditions on the ranges. In [6], the same author studied the transmission of some properties from a bounded linear operator, as closedness of range and generalized inverses, to its restriction on certain subspaces and vice-versa. On the other hand, it is well known that, if two operators are similar (see [1]) then their spectra are equals, and that this equality extends to several finer structures of the spectra as point spectra, approximate point spectrum, Fredholm points, etc. Motivated by these researches, in this paper we continue investigating the behavior of several spectra derived from the classical Fredholm theory for

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an operator T and its restriction T_W on a proper closed and T-invariant subspace $W \subseteq X$ such that $T^n(X) \subseteq W$ for some $n \geqslant 1$, where $T \in L(X)$ and X is an infinite-dimensional complex Banach space. The main goal of this paper is to study the relationship between the spectra derived from Fredholm theory corresponding to T and T_W, in order to obtain sufficient conditions for which the spectra derived from Fredholm theory and other parts of the spectra corresponding to two given operators are preserved. As an application of our results, we give conditions for which the above mentioned spectra corresponding to two multiplication operators acting on the space of functions of bounded p-variation in Wiener's sense coincide. Some additional illustrative results are given too.

2. Preliminaries

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. The classes of operators studied in the classical Fredholm theory generate several spectra associated with an operator $T \in L(X)$. The *Fredholm spectrum* is defined by

$$\sigma_{\rm f}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not Fredholm}\},$$

the *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\rm uf}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not upper semi-Fredholm}\},$$

and the *lower semi-Fredholm spectrum* is defined by

$$\sigma_{\rm lf}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not lower semi-Fredholm}\}.$$

The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_{\rm b}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not Browder}\},$$

and

$$\sigma_{\rm w}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not Weyl}\}.$$

Since every Browder operator is Weyl, $\sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T)$. Analogously, the *upper semi-Browder spectrum* and the *upper semi-Weyl spectrum* are defined by

$$\sigma_{\rm ub}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not upper semi-Browder}\},$$

and

$$\sigma_{\mathrm{uw}}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not upper semi-Weyl}\}.$$

Similarly, the *lower semi-Browder spectrum* and the *lower semi-Weyl spectrum* are defined by

$$\sigma_{\mathrm{lb}}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not lower semi-Browder}\},$$

and

$$\sigma_{\mathrm{lw}}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not lower semi-Weyl}\}.$$

For further information on Fredholm operators theory, we refer to [1] and [11].

Another important class of operators is the quasi-Fredholm operators defined in the sequel. First, we consider the set

$$\Delta(T) = \{n \in \mathbb{N}\colon\ m \geqslant n,\, m \in \mathbb{N} \Rightarrow T^n(X) \cap N(T) \subseteq T^m(X) \cap N(T)\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) = \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.1. An operator $T \in L(X)$ is said to be *quasi-Fredholm of degree* d, if there exists $d \in \mathbb{N}$ such that:
(a) $\mathrm{dis}(T) = d$,
(b) $T^n(X)$ is a closed subspace of X for each $n \geqslant d$,
(c) $T(X) + N(T^d)$ is a closed subspace of X.

For further information on quasi-Fredholm operators, we refer to [2], [3], [7] and [8].

Definition 2.2 ([10])**.** An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at λ_0), if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f\colon \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \forall\, \lambda \in \mathbb{D}_{\lambda_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\varrho(T) = \mathbb{C} \setminus \sigma(T)$. Also, the SVEP is inherited by restrictions on invariant closed subspaces. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [1], Theorem 3.8)

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$(2)\qquad q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda.$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_{\rm ap}(T)$ the classical *approximate point spectrum* defined by

$$\sigma_{\rm ap}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not bounded below}\}.$$

Note that if $\sigma_{\rm su}(T)$ denotes the *surjectivity spectrum*

$$\sigma_{\rm su}(T) = \{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not onto}\},$$

then $\sigma_{\rm ap}(T) = \sigma_{\rm su}(T^*)$, $\sigma_{\rm su}(T) = \sigma_{\rm ap}(T^*)$ and $\sigma(T) = \sigma_{\rm ap}(T) \cup \sigma_{\rm su}(T)$.

It is easily seen from the definition of localized SVEP that

$$(3)\qquad \lambda \notin \operatorname{acc} \sigma_{\rm ap}(T) \Rightarrow T \text{ has SVEP at } \lambda,$$

and

$$(4)\qquad \lambda \notin \operatorname{acc} \sigma_{\rm su}(T) \Rightarrow T^* \text{ has SVEP at } \lambda,$$

where $\operatorname{acc} K$ means the set of all accumulation points of a subset $K \subseteq \mathbb{C}$.

Remark 2.3. The implications (1), (2), (3) and (4) are actually equivalences, if $T \in L(X)$ is semi-Fredholm (see [1], Chapter 3). More generally, if $T \in L(X)$ is quasi-Fredholm (see [2]). On the other hand, $\sigma_{\rm b}(T) = \sigma_{\rm w}(T) \cup \operatorname{acc} \sigma(T)$, $\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T) \cup \operatorname{acc} \sigma_{ap}(T)$ and $\sigma(T) = \sigma_{\rm ap}(T) \cup \Xi(T)$, where $\Xi(T)$ denotes the set $\{\lambda \in \mathbb{C}\colon$ T does not have SVEP at $\lambda\}$ (see [1], Chapter 3).

According to the notation of Barnes [6], in the sequel of this paper we always assume that W is a proper closed subspace of a Banach space X. Also, we denote

$$\mathcal{P}(X, W) = \{T \in L(X)\colon\ T(W) \subseteq W \text{ and for some integer } n \geqslant 1,\ T^n(X) \subseteq W\}.$$

For each $T \in \mathcal{P}(X, W)$, T_W denotes the restriction of T on the T-invariant subspace W of X. Observe that $0 \in \sigma_{\rm su}(T)$ for all $T \in \mathcal{P}(X, W)$. Because, $T \in \mathcal{P}(X, W)$ and T onto implies that $X = T^n(X) \subseteq W$ for some $n \geqslant 1$, contradicting our assumption that W is a proper subspace of X. Later we shall see that $\sigma_{\rm su}(T)$ and $\sigma_{\rm su}(T_W)$ may differ only in 0.

R e m a r k 2.4. Observe that an operator $F \in L(W)$ with n-dimensional range has the form $F(w) = \sum\limits_{k=1}^{n} f_k(w)F(w_k)$, where $F(w_k) \in W$ and $f_k \in W^*$ (W^* denotes the dual space of W) for $k = 1, \ldots, n$. By the Hahn-Banach Theorem, each $f_k \in W^*$ has an extension $\hat{f}_k \in X^*$ (X^* denotes the dual space of X), then F has an extension $\widehat{F} \in L(X)$, with finite-dimensional range, given by $\widehat{F}(x) = \sum\limits_{k=1}^{n} \hat{f}_k(x)F(w_k)$ for all $x \in X$. Also, $\widehat{F} \in \mathcal{P}(X, W)$ and $\widehat{F}_W = F$.

We end this section by stating the following lemmas which were proved in [6].

Lemma 2.5 ([6], Proposition 3)**.** *Let $T \in \mathcal{P}(X, W)$. Then $(\lambda I - T)^{-1}(W) = W$ for all $\lambda \neq 0$.*

Lemma 2.6 ([6], Theorem 6 (1))**.** *Let $T \in \mathcal{P}(X, W)$. Then for all $\lambda \neq 0$, we have*

$$R(\lambda I - T) \text{ is closed in } X \text{ if and only if } R(\lambda I - T_W) \text{ is closed in } W.$$

3. Basic relations between the spectra of T and T_W

In this section, we establish several lemmas that will be used throughout this paper. These lemmas describe some important relations between an operator $T \in \mathcal{P}(X, W)$ and its restriction T_W.

We begin by extending the basic equality $N(\lambda I - T) = T(N(\lambda I - T))$ for $\lambda \neq 0$, as follow.

Lemma 3.1. *Let $T \in L(X)$. Then $N((\lambda I - T)^m) = T^n(N((\lambda I - T)^m))$ for all $\lambda \neq 0$ and any $n, m \in \mathbb{N}$.*

P r o o f. It follows by mathematical induction. □

The next lemma is a generalization of [9], Lemma 2.1, but in the framework dealt with by Barnes in [6].

Lemma 3.2. *If $T \in \mathcal{P}(X, W)$, then for all $\lambda \neq 0$:*

(i) $N((\lambda I - T_W)^m) = N((\lambda I - T)^m)$ *for any* m,
(ii) $R((\lambda I - T_W)^m) = R((\lambda I - T)^m) \cap W$ *for any* m,
(iii) $\alpha(\lambda I - T_W) = \alpha(\lambda I - T)$,
(iv) $p(\lambda I - T_W) = p(\lambda I - T)$,
(v) $\beta(\lambda I - T_W) = \beta(\lambda I - T)$.

P r o o f. The proof is similar to that of [9], Lemma 2.1, making use of Lemma 3.1 in part (i) and Lemma 2.5 in part (ii). □

Moreover, we have the following equivalences.

Lemma 3.3. *If $T \in \mathcal{P}(X, W)$, then:*

(i) *$p(T) < \infty$ if and only if $p(T_W) < \infty$,*

(ii) *$q(T) < \infty$ if and only if $q(T_W) < \infty$.*

P r o o f. (i) Since T_W is a restriction of T on the subspace T-invariant W of X, then $N(T_W^k) = N(T^k) \cap W$ for all $k \in \mathbb{N}$. In consequence, $p(T) < \infty$ implies that $p(T_W) < \infty$. Reciprocally, by [11], Proposition 38.1, $p(T_W) < \infty$ implies that $T_W^m(W) \cap N(T_W^k) = \{0\}$ for every integer $m \geqslant p(T_W)$ and every natural number k. Also, if $T \in \mathcal{P}(X, W)$ there exists $n \geqslant 1$ such that $T^n(X) \subseteq W$. Hence $T^{n+m}(X) \subseteq T^m(W) \subseteq W$ for all $m \in \mathbb{N}$, thus

$$\{0\} \subseteq T^{m+n}(X) \cap N(T^k) \subseteq T^m(W) \cap W \cap N(T^k) = T_W^m(W) \cap N(T_W^k) = \{0\}.$$

Therefore $T^{m+n}(X) \cap N(T^k) = \{0\}$ for any $m, k \in \mathbb{N}$. Again, by [11], Proposition 38.1, $p(T) \leqslant m + n < \infty$.

(ii) As observed in (i), $T^{m+n}(X) \subseteq T^m(W)$ for all $m \in \mathbb{N}$. Then

$$T_W^{m+n}(W) = T^{m+n}(W) \subseteq T^{m+n}(X) \subseteq T^m(W) = T_W^m(W) \subseteq T^m(X)$$

for all $m \in \mathbb{N}$, from which we deduce that $q(T) < \infty$ if and only if $q(T_W) < \infty$. □

In the same style as in the Lemma 2.6, the following result treats the relationship between the SVEP of an operator $T \in \mathcal{P}(X, W)$ and its restriction T_W.

Lemma 3.4. *If $T \in \mathcal{P}(X, W)$, then T has SVEP at λ if and only if T_W has SVEP at λ.*

P r o o f. It is easy to see that T, respectively, T_W has the SVEP at λ if and only if $\lambda I - T$, respectively, $\lambda I - T_W$ has the SVEP at 0. Thus, we may assume without loss of generality $\lambda = 0$. Since the SVEP is inherited by restrictions on invariant closed subspaces, if T has the SVEP at 0 then T_W has the SVEP at 0. Reciprocally, suppose that T_W has the SVEP at 0 and let us consider an open disc $\mathbb{D}_0 \subseteq \mathbb{C}$ centered at 0 and an analytic function $f\colon \mathbb{D}_0 \to X$ such that $(\mu I - T)f(\mu) = 0$ for all $\mu \in \mathbb{D}_0$. This implies that $\mu^k f(\mu) = T^k f(\mu)$ for all $k \in \mathbb{N}$. Consequently, since $T \in \mathcal{P}(X, W)$, there exists $n \geqslant 1$ such that $T^n(X) \subseteq W$ and so $f(\mu) = \mu^{-n} T^n f(\mu) \in T^n(X) \subseteq W$ for all $\mu \in \mathbb{D}_0 \setminus \{0\}$. On the other hand, if $\mu = 0$ there exists a sequence $(\lambda_k)_{k=1}^{\infty} \subseteq \mathbb{D}_0$

such that $\lambda_k \neq 0$ and $\lambda_k \to 0$. Hence, $(f(\lambda_k))_{k=1}^{\infty} \subseteq W$ and $f(\lambda_k) \to f(0)$. Being W a closed subspace, we conclude that $f(0) \in W$. Therefore $f\colon \mathbb{D}_0 \to W$ is an analytic function such that $(\mu I - T_W)f(\mu) = 0$ for every $\mu \in \mathbb{D}_0$. From this, by the assumption that T_W has the SVEP at 0, we deduce that $f \equiv 0$ on $\mathbb{D}_0$ and therefore T has the SVEP at 0. □

4. Main results and applications

In this section we present the main results and applications of this paper. We give sufficient conditions for the spectra derived from the Fredholm theory and other parts of the spectra corresponding to two given operators to be preserved. Applications to multiplication operators acting on the space of functions of bounded p-variation in Wiener's sense are given. Additional illustrative results are given too.

The following result treats the spectral relationships between the operator $T \in \mathcal{P}(X, W)$ and its restriction T_W for several spectra derived from the classical Fredholm theory.

Theorem 4.1. *If $T \in \mathcal{P}(X, W)$ and $q(T) = \infty$, or $p(T) = \infty$, then the following equalities are true:*

(i) $\sigma_{\rm su}(T) = \sigma_{\rm su}(T_W)$;

(ii) $\sigma_{\rm ap}(T) = \sigma_{\rm ap}(T_W)$;

(iii) $\sigma(T) = \sigma(T_W)$;

(iv) $\sigma_{\rm w}(T) = \sigma_{\rm w}(T_W)$;

(v) $\sigma_{\rm uw}(T) = \sigma_{\rm uw}(T_W)$;

(vi) $\sigma_{\rm b}(T) = \sigma_{\rm b}(T_W)$;

(vii) $\sigma_{\rm ub}(T) = \sigma_{\rm ub}(T_W)$;

(viii) $\sigma_{\rm f}(T) = \sigma_{\rm f}(T_W)$;

(ix) $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(T_W)$.

Proof. (i) Observe first that $\lambda I - T$, respectively, $\lambda I - T_W$ is onto if and only if $\beta(\lambda I - T) = 0$, respectively, $\beta(\lambda I - T_W) = 0$. Now, by Lemma 3.2, $\beta(\lambda I - T) = \beta(\lambda I - T_W)$ for all $\lambda \neq 0$, and then $\sigma_{\rm su}(T) \setminus \{0\} = \sigma_{\rm su}(T_W) \setminus \{0\}$. To show the equality $\sigma_{\rm su}(T) = \sigma_{\rm su}(T_W)$ we need only to prove that $0 \in \sigma_{\rm su}(T) \cap \sigma_{\rm su}(T_W)$. Since $T \in \mathcal{P}(X, W)$, $0 \in \sigma_{\rm su}(T)$. We claim that $0 \in \sigma_{\rm su}(T_W)$. To see this, suppose that $0 \notin \sigma_{\rm su}(T_W)$. Then T_W is onto, thus $W = (T_W)^k(W) = T^k(W)$ for $k = 0, 1, 2, \ldots$ Being $T \in \mathcal{P}(X, W)$, there exist $n \geqslant 1$ such that $T^n(X) \subseteq W$, then $W = T^m(W) \subseteq T^m(X) \subseteq T^n(X) \subseteq W$ for all $m \geqslant n$. Therefore $T^m(X) = T^n(X) = T^m(W) = W$ for all $m \geqslant n$, which implies that $q(T) < \infty$, contradicting our assumption that

$q(T)=\infty$. On the other hand, T_W onto implies that $q(T_W)=0$, and so $(T_W)^*$ has the SVEP at 0. Hence, $0\notin\Xi((T_W)^*)$. From this,

$$0\notin\sigma_{\mathrm{su}}(T_W)\cup\Xi((T_W)^*)=\sigma((T_W)^*)=\sigma(T_W)=\sigma_{\mathrm{ap}}(T_W)\cup\Xi(T_W).$$

Consequently $0\notin\Xi(T_W)$, that is, T_W has the SVEP at 0. Since $T\in\mathcal{P}(X,W)$, by Lemma 3.4, T has the SVEP at 0. But, as observed above, $T\in\mathcal{P}(X,W)$ implies that there exists $n\geqslant 1$ such that $T^m(X)=T^n(W)=W$ for all $m\geqslant n$. Then, by the isomorphism $T^k(X)/T^{k+1}(X)\cong X/N(T^k)+T(X)$ (for all $k\in\mathbb{N}$), given by $T^kx+T^{k+1}(X)\to x+(N(T^k)+T(X))$, we conclude that $X=N(T^m)+T(X)$ for all $m\geqslant n$. Also $\mathrm{dis}(T)=\inf\Delta(T)\leqslant n$, because $T^m(X)\cap N(T)=T^n(X)\cap N(T)$ for all $m\geqslant n$. Thus, T is a quasi-Fredholm operator and T has the SVEP at 0. By [2], Theorem. 2.7, $p(T)<\infty$, contradicting our assumption that $p(T)=\infty$.

(ii) Note first that for each $\lambda\in\sigma_{\mathrm{ap}}(T)\setminus\{0\}$, $\lambda I-T$ is not bounded below and $\lambda\neq 0$. Therefore, we have the following possibilities: $p(\lambda I-T)>0$ or $R(\lambda I-T)$ is not closed in X. But, by Lemmas 3.2 and 2.6, these possibilities are equivalent to $p(\lambda I-T_W)>0$ or $R(\lambda I-T_W)$ is not closed in W. Hence $\sigma_{\mathrm{ap}}(T)\setminus\{0\}=\sigma_{\mathrm{ap}}(T_W)\setminus\{0\}$. As in part (i), for the equality $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{ap}}(T_W)$, it suffices to show that $0\in\sigma_{\mathrm{ap}}(T)\cap\sigma_{\mathrm{ap}}(T_W)$. Suppose that $0\notin\sigma_{\mathrm{ap}}(T)$ then T is injective. Consequently T has SVEP at 0, then $0\notin\Xi(T)$. But, since $\sigma_{\mathrm{ap}}(T)\cup\Xi(T)=\sigma(T)=\sigma_{\mathrm{ap}}(T)\cup\sigma_{\mathrm{su}}(T)$, we have that $0\notin\sigma_{\mathrm{su}}(T)$, a contradiction. Therefore $0\in\sigma_{\mathrm{ap}}(T)$. Similarly, $0\notin\sigma_{\mathrm{ap}}(T_W)$ implies T_W injective. Thus, T_W has SVEP at 0 and $0\notin\Xi(T_W)$. Again, since $\sigma_{\mathrm{ap}}(T_W)\cup\Xi(T_W)=\sigma(T_W)=\sigma_{\mathrm{ap}}(T_W)\cup\sigma_{\mathrm{su}}(T_W)$, we have that $0\notin\sigma_{\mathrm{su}}(T_W)$. By part (i), $0\notin\sigma_{\mathrm{su}}(T)$, and as observed above this is impossible. Then $0\in\sigma_{\mathrm{ap}}(T_W)$, so the equality $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{ap}}(T_W)$ holds.

(iii) To show the equality $\sigma(T)=\sigma(T_W)$, observe that $\sigma(T)=\sigma_{\mathrm{ap}}(T)\cup\sigma_{\mathrm{su}}(T)$ (or $\sigma(T_W)=\sigma_{\mathrm{ap}}(T_W)\cup\sigma_{\mathrm{su}}(T_W)$). Hence, combining these equalities with (i) and (ii), we obtain that $\sigma(T)=\sigma(T_W)$.

(iv) Proceeding as in the first part of proofs (i) and (ii), by Lemmas 3.2 and 2.6, we see that $\sigma_{\mathrm{f}}(T)\setminus\{0\}=\sigma_{\mathrm{f}}(T_W)\setminus\{0\}$ and $\sigma_{\mathrm{w}}(T)\setminus\{0\}=\sigma_{\mathrm{w}}(T_W)\setminus\{0\}$. Again, as in parts (i) and (ii), for the equality $\sigma_{\mathrm{w}}(T)=\sigma_{\mathrm{w}}(T_W)$ it suffices to show that $0\in\sigma_{\mathrm{w}}(T)\cap\sigma_{\mathrm{w}}(T_W)$. Note first that, if $0\notin\sigma_{\mathrm{w}}(T)$ then T is a Weyl operator. That is, T is a Fredholm operator with $\mathrm{ind}(T)=0$. Being $T\in\mathcal{P}(X,W)$, there exists $n\geqslant 1$ such that $T^n(X)\subseteq W$, from which we obtain the inclusions

$$T^{n+m}(X)\subseteq T^m(W)\subseteq W\subseteq X\quad\forall\, m\in\mathbb{N},$$

and so the inequalities

$$\dim\frac{W}{T^{n+m}(X)}\geqslant\dim\frac{W}{T^m(W)}=\beta(T_W^m).$$

Since $W/T^{n+m}(X) \subseteq X/T^{n+m}(X)$, we have

$$\beta(T^{n+m}) = \dim \frac{X}{T^{n+m}(X)} \geqslant \dim \frac{W}{T^{n+m}(X)} \geqslant \dim \frac{W}{T^{m}(W)} = \beta(T_W^m).$$

Thus, $\beta(T^{n+m}) \geqslant \beta(T_W^m)$ for any $m \in \mathbb{N}$. On the other hand, the inclusions $N(T_W^m) \subseteq N(T^m) \subseteq N(T^{n+m})$, imply $\alpha(T_W^m) \leqslant \alpha(T^{n+m})$. Then $T_W^m \in L(W)$ is a Fredholm operator, so T_W is a Fredholm operator. Since $T \in L(X)$ is a Weyl operator, by [11], Proposition 26.2, there exists a bijective operator $R \in L(X)$ and a finite rank operator $K \in L(X)$ such that $T = R + K$. Therefore $T_W = R_W + K_W$, with R_W injective and K_W of finite rank. This yields that

$$\operatorname{ind}(T_W) = \operatorname{ind}(R_W + K_W) = \operatorname{ind}(R_W) \leqslant 0.$$

Thus, we conclude that $T_W \in L(W)$ is a upper semi-Weyl operator. Again, by [11], Proposition 26.2, there exists a injective operator $S \in L(W)$ and a finite rank operator $F \in L(W)$ such that $T_W = S + F$, from which $S = T_W - F$. But, since $T_W(W)$ is closed and $F(W)$ is a finite dimensional subspace of W, $S(W)$ is closed in W. So $S \in L(W)$ is bounded below, and hence $0 \notin \sigma_{\mathrm{ap}}(S) = \sigma_{\mathrm{ap}}(T_W - F)$. By Remark 2.4, $F \in L(W)$ has an extension $\widehat{F} \in L(X)$ such that $\widehat{F} \in \mathcal{P}(X, W)$, then $T - \widehat{F} \in \mathcal{P}(X, W)$. Consequently, $(T - \widehat{F})_W = T_W - F$. Thus, by part (ii), $0 \in \sigma_{\mathrm{ap}}(T - \widehat{F}) = \sigma_{\mathrm{ap}}((T - \widehat{F})_W) = \sigma_{\mathrm{ap}}(T_W - F)$. That is, $0 \in \sigma_{\mathrm{ap}}(T_W - F)$ and $0 \notin \sigma_{\mathrm{ap}}(T_W - F)$, a contradiction. Hence $0 \in \sigma_{\mathrm{w}}(T)$. Now, we show that $0 \in \sigma_{\mathrm{w}}(T_W)$. To see this, suppose that $0 \notin \sigma_{\mathrm{w}}(T_W) = \sigma_{\mathrm{uw}}(T_W) \cup \sigma_{\mathrm{lw}}(T_W)$. It follows that $0 \notin \sigma_{\mathrm{uw}}(T_W)$. That is, $T_W \in L(W)$ is an upper semi-Weyl operator. But, as observed above this is impossible, hence $0 \in \sigma_{\mathrm{w}}(T_W)$. Consequently, we obtain the equality $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T_W)$.

(v) Again, as in the first part of proofs (i) and (ii), by Lemmas 3.2 and 2.6, we have that $\sigma_{\mathrm{uf}}(T) \setminus \{0\} = \sigma_{\mathrm{uf}}(T_W) \setminus \{0\}$ and $\sigma_{\mathrm{uw}}(T) \setminus \{0\} = \sigma_{\mathrm{uw}}(T_W) \setminus \{0\}$. As in the proof of part (iv), to show the equality $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T_W)$ we need only to prove that $0 \in \sigma_{\mathrm{uw}}(T) \cap \sigma_{\mathrm{uw}}(T_W)$. By similar representation arguments for semi-Weyl operators as in part (iv), we can prove that $0 \in \sigma_{\mathrm{uw}}(T_W)$ and $0 \in \sigma_{\mathrm{uw}}(T)$.

Finally, to show parts (vi) and (vii), observe that $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{acc} \sigma(T)$ and $\sigma_{\mathrm{b}}(T_W) = \sigma_{\mathrm{w}}(T_W) \cup \operatorname{acc} \sigma(T_W)$. Hence, combining these equalities with (iii) and (iv), we obtain that $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(T_W)$. Similarly, combining the equalities $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{acc} \sigma_{\mathrm{ap}}(T)$ and $\sigma_{\mathrm{ub}}(T_W) = \sigma_{\mathrm{uw}}(T_W) \cup \operatorname{acc} \sigma_{\mathrm{ap}}(T_W)$ with (ii) and (v) yields $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T_W)$.

(viii) As observed in (iv), if $T \in \mathcal{P}(X, W)$ there exists $n \geqslant 1$ such that $T^n(X) \subseteq W$ and the inclusions

$$T^{n+m}(X) \subseteq T^m(W) \subseteq W \subseteq X \quad \forall\, m \in \mathbb{N}$$

hold. This implies that,

$$\dim \frac{W}{T^{n+m}(X)} \geqslant \dim \frac{W}{T^{m}(W)} = \beta(T_W^m).$$

But, since $W/T^{n+m}(X) \subseteq X/T^{n+m}(X)$, we have

$$\beta(T^{n+m}) = \dim \frac{X}{T^{n+m}(X)} \geqslant \dim \frac{W}{T^{n+m}(X)} \geqslant \dim \frac{W}{T^{m}(W)} = \beta(T_W^m).$$

Thus, $\beta(T^{n+m}) \geqslant \beta(T_W^m)$ for any $m \in \mathbb{N}$. Also, $N(T_W^m) \subseteq N(T^m) \subseteq N(T^{n+m})$ implies that $\alpha(T_W^m) \leqslant \alpha(T^{n+m})$. Consequently, $T_W \in L(W)$ is a Fredholm operator if $T \in L(X)$ is a Fredholm operator. Reciprocally, if $T_W \in L(W)$ is a Fredholm operator, then $T_W^m \in L(W)$ is a Fredholm operator for every non-negative integer m. In particular, for $n \geqslant 1$ such that $T^n(X) \subset W$, $T_W^n \in L(W)$ is Fredholm. Therefore there exists an operator $S \in L(W)$ and a finite rank operator $F \in L(W)$ such that $ST_W^n - F$ is the identity on W. Consider $P = ST^n - \widehat{F}$, $\widehat{F}$ given by Remark 2.4, this function is a bounded projection of X onto W. That is, $P \in L(X)$, $P^2 = P$ and $P(X) = W$. Let $T_P\colon P(X) \to P(X)$, $T_P y = TPy$, the compression of T generated from P. Since $X = N(P) \oplus P(X)$ and $(I-P)(X) = N(P) \subseteq (ST^n)^{-1}(\widehat{F}(X))$ is finite-dimensional, $\alpha(P) = \beta(P) < \infty$. It is easily seen that $\alpha(T_P) = \alpha(PTP) - \alpha(P)$ and $\beta(T_P) = \beta(PTP) - \beta(P)$. On the other hand, since $P(X) = W$ is a T-invariant subspace, then $T_P = T_W$. Thus we have the equalities $\alpha(T_W) = \alpha(PTP) - \alpha(P)$ and $\beta(T_W) = \beta(PTP) - \beta(P)$, from which it follows that both $\alpha(PTP)$ and $\beta(PTP)$ are finite. That is, $PTP \in L(X)$ is a Fredholm operator. Moreover, since

$$T = PT + (I-P)T = PTP + PT(I-P) + (I-P)T,$$

$T \in L(X)$ is Fredholm if and only if $PTP \in L(X)$ is Fredholm, because $T - PTP = PT(I-P) + (I-P)T$ and $PT(I-P) + (I-P)T$ is a finite rank operator in $L(X)$. In consequence, $T \in L(X)$ is a Fredholm operator. Thus we have proved that

$$T \text{ is Fredhom in } L(X) \Rightarrow T_W \text{ is Fredhom in } L(W) \Rightarrow T \text{ is Fredhom in } L(X).$$

But, by Lemmas 3.2 and 2.6, $\sigma_{\rm f}(T) \setminus \{0\} = \sigma_{\rm f}(T_W) \setminus \{0\}$. Then, we can conclude that $\sigma_{\rm f}(T) = \sigma_{\rm f}(T_W)$.

(ix) The proof is analogous to that of part (viii) applying representation theorems for upper semi-Fredholm operators. □

R e m a r k 4.2. Recall that for $T \in L(X)$, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T (see [11], Proposition 50.2). Also, it is well known that if λ is a pole of the resolvent of T, then $\lambda \in \operatorname{iso} \sigma(T)$. Evidently, if $\lambda \in \operatorname{iso} \sigma(T)$ then $\lambda \in \partial\sigma(T)$. Thus, for $T \in \mathcal{P}(X, W)$, if $0 \notin \operatorname{iso} \sigma(T)$ (or $0 \notin \partial\sigma(T)$, $0 \in \Xi(T)$, $0 \in \Xi(T^*)$) then $q(T) = \infty$ or $p(T) = \infty$. Therefore, the conclusions of Theorem 4.1 remain true if the hypothesis $q(T) = \infty$ or $p(T) = \infty$ is replaced by one of the following hypotheses: $0 \notin \operatorname{iso} \sigma(T)$, $0 \notin \partial\sigma(T)$, $0 \in \Xi(T)$ or $0 \in \Xi(T^*)$.

Remark 4.3. According to Lemma 3.3 we can change the hypothesis $p(T) = \infty$ or $q(T) = \infty$ by $p(T_W) = \infty$ or $q(T_W) = \infty$ in Theorem 4.1. Consequently, by the above remark, the conclusions of Theorem 4.1 remain true if the hypothesis $p(T) = \infty$, or $q(T) = \infty$, is replaced by one of the following hypotheses: $0 \notin \operatorname{iso} \sigma(T_W)$, $0 \notin \partial\sigma(T_W)$, $0 \in \Xi(T_W)$ or $0 \in \Xi(T_W^*)$.

We give an illustrative example for the behavior of the spectra of an operator T and its restriction T_W, when T does not satisfy the hypothesis of Theorem 4.1.

Example 4.4. Let X be a Banach space, and assume that W and Z are proper closed subspaces of X with $X = W \oplus Z$. Let T be the projection of X on W which is zero on Z. Since T is a projection operator, i.e. $T^2 = T$, hence $\sigma(T) = \{0, 1\}$. Moreover, $\sigma_{\rm su}(T) = \sigma_{\rm ap}(T) = \sigma_{\rm w}(T) = \sigma_{\rm uw}(T) = \sigma_{\rm b}(T) = \sigma_{\rm ub}(T) = \sigma(T)$. On the other hand, the operator $T_W = T|_{T(X)}$ is the identity operator on W, so $\sigma(T_W) = \{1\}$. Also, $\sigma_{\rm su}(T_W) = \sigma_{\rm ap}(T_W) = \sigma_{\rm w}(T_W) = \sigma_{\rm uw}(T_W) = \sigma_{\rm b}(T_W) = \sigma_{\rm ub}(T_W) = \sigma(T_W)$.

As an immediate application of Theorem 4.1 and Remark 4.2, we obtain sufficient conditions for the Fredholm properties corresponding to two given operators to coincide.

Theorem 4.5. *Suppose that $T, S \in \mathcal{P}(X, W)$ and T, S coincide on W. Let one of the following conditions is valid:*

(i) $0 \notin \operatorname{iso} \sigma(T) \cup \operatorname{iso} \sigma(S)$,
(ii) $0 \notin \partial\sigma(T) \cup \partial\sigma(S)$,
(iii) $0 \in \Xi(T) \cap \Xi(S)$,
(iv) $0 \in \Xi(T^*) \cap \Xi(S^*)$.

Then the following equalities are true:

(i) $\sigma_{\rm su}(T) = \sigma_{\rm su}(S)$, $\sigma_{\rm ap}(T) = \sigma_{\rm ap}(S)$ *and* $\sigma(T) = \sigma(S)$.
(ii) $\sigma_{\rm w}(T) = \sigma_{\rm w}(S)$ *and* $\sigma_{\rm uw}(T) = \sigma_{\rm uw}(S)$.
(iii) $\sigma_{\rm b}(T) = \sigma_{\rm b}(S)$ *and* $\sigma_{\rm ub}(T) = \sigma_{\rm ub}(S)$.
(iv) $\sigma_{\rm f}(T) = \sigma_{\rm f}(S)$ *and* $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(S)$.

Proof. The given theorem immediately follows from Theorem 4.1 and Remark 4.2, since $T_W = S_W$. □

As a consequence of Theorem 4.5 and Remark 4.3, we obtain additional conditions under which the Fredholm properties corresponding to two given operators coincide.

Corollary 4.6. *Suppose that* $T, S \in \mathcal{P}(X, W)$ *and* T, S *coincide on* W. *Let one of the following conditions is valid:*

(i) $0 \notin \operatorname{iso} \sigma(T_W)$ (*or* $0 \notin \operatorname{iso} \sigma(S_W)$),
(ii) $0 \notin \partial\sigma(T_W)$ (*or* $0 \notin \partial\sigma(S_W)$),
(iii) $0 \in \Xi(T_W)$ (*or* $0 \in \Xi(S_W)$),
(iv) $0 \in \Xi(T_W^*)$ (*or* $0 \in \Xi(S_W^*)$).

Then the following equalities are true:

(i) $\sigma_{\rm su}(T) = \sigma_{\rm su}(S)$, $\sigma_{\rm ap}(T) = \sigma_{\rm ap}(S)$ *and* $\sigma(T) = \sigma(S)$.
(ii) $\sigma_{\rm w}(T) = \sigma_{\rm w}(S)$ *and* $\sigma_{\rm uw}(T) = \sigma_{\rm uw}(S)$.
(iii) $\sigma_{\rm b}(T) = \sigma_{\rm b}(S)$ *and* $\sigma_{\rm ub}(T) = \sigma_{\rm ub}(S)$.
(iv) $\sigma_{\rm f}(T) = \sigma_{\rm f}(S)$ *and* $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(S)$.

P r o o f. The corollary immediately follows from Theorem 4.5 and Remark 4.3. □

Astudillo-Villaba, Castillo and Ramos-Fernández in [4] studied invertibility, compactness and closedness of the range for multiplication operators acting on the space of functions of bounded p-variation in Wiener's sense $\mathrm{W\,BV}_p[0,1]$. We give some applications of our results for this class of operators.

Corollary 4.7. *Let* $\mathrm{W\,BV}_p[0,1]$ *be the space of functions of bounded* p*-variation in Wiener's sense on* $[0,1]$. *Suppose that* $u \in \mathrm{W\,BV}_p[0,1]$ *and consider the multiplication operator induced by* u, $M_u\colon \mathrm{W\,BV}_p[0,1] \to \mathrm{W\,BV}_p[0,1]$, *given by* $M_u(f) = u\cdot f$. *If the set* Z_u *of zeros of* u *in* $[0,1]$ *is a nonempty set and* $0 \notin \operatorname{iso} u([0,1])$, *then all spectral equalities* (i)–(ix) *of Theorem* 4.1 *for* M_u *and its restriction on the subspace* $X_{Z_u} = \{f \in \mathrm{W\,BV}_p[0,1]\colon f(t) = 0$ *for all* $t \in Z_u\}$ *are true.*

P r o o f. If $Z_u \neq \emptyset$, by [4], Proposition 6, X_{Z_u} is a proper closed M_u-invariant subspace of $\mathrm{W\,BV}_p[0,1]$ such that $M_u(\mathrm{W\,BV}_p[0,1]) \subset X_{Z_u}$. That is, $M_u \in \mathcal{P}(\mathrm{W\,BV}_p[0,1], X_{Z_u})$. On the other hand, if both $p(M_u)$ and $q(M_u)$ are finite then $0 < p(M_u) = q(M_u) < \infty$. So, as observed in Remark 4.2, 0 is a pole of the resolvent of M_u and hence $0 \in \operatorname{iso} \sigma(M_u) = \operatorname{iso} \overline{(u[0,1])}$, a contradiction. Thus, $p(M_u) = \infty$ or $q(M_u) = \infty$. Therefore by Theorem 4.1, we can conclude that all spectral equalities (i)–(ix) of Theorem 4.1 for M_u and its restriction on the subspace X_{Z_u} are true. □

Corollary 4.8. *If* $u, v \in \mathrm{WBV}_p[0,1]$ *have the same zeros in* $[0,1]$, *and* $0 \notin \mathrm{iso}\, u([0,1]) \cup \mathrm{iso}\, v([0,1])$, *then all spectral equalities* (i)–(iv) *of Theorem* 4.5 *for* M_u *and* M_v *are true.*

P r o o f. Suppose that $u, v \in \mathrm{WBV}_p[0,1]$ have the same zeros in $[0,1]$. Then $Z_u = Z_v$, so $X_{Z_u} = X_{Z_v}$. Also $0 \notin \mathrm{iso}\,\sigma(M_u) \cup \mathrm{iso}\,\sigma(M_v)$, because $0 \notin \mathrm{iso}\, u([0,1]) \cup \mathrm{iso}\, v([0,1])$. Taking $W = X_{Z_u} = X_{Z_v}$, by Theorem 4.5 we have that all spectral equalities (i)–(iv) of Theorem 4.5 for M_u and M_v are true. $\square$

R e m a r k 4.9. It is well known that, if two operators are similar then their spectra are equals, and that this equality extends to several finer structures of the spectra as point spectra, approximate point spectrum, Fredholm points, etc. Here we study this situation, where the notion of similar operators is replaced by the simplest hypotheses. Results analogous to Corollaries 4.7 and 4.8, can be proved for composition operators and integral operators by using our results.

As a final application of our results, we state the following theorem which ensures that bounded operators acting on complemented subspaces can always be extended to the entire space preserving spectral properties.

Theorem 4.10. *Let* W *be a complemented subspace of* X *and* $T \in L(W)$. *If one of the following conditions is valid:*

(i) $0 \notin \mathrm{iso}\,\sigma(T)$,
(ii) $0 \notin \partial\sigma(T)$,
(iii) $0 \in \Xi(T)$,
(iv) $0 \in \Xi(T^*)$,

then T *has an extension* $\overline{T} \in \mathcal{P}(X, W)$ *and the following equalities are true:*

(i) $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{su}}(\overline{T})$, $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(\overline{T})$ *and* $\sigma(T) = \sigma(\overline{T})$,
(ii) $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(\overline{T})$ *and* $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(\overline{T})$,
(iii) $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(\overline{T})$ *and* $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(\overline{T})$,
(iv) $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{f}}(\overline{T})$ *and* $\sigma_{\mathrm{uf}}(T) = \sigma_{\mathrm{uf}}(\overline{T})$.

P r o o f. Since W is a complemented subspace of X, there exists a bounded projection $P \in L(X)$ such that $P(X) = W$. Thus $\overline{T} = TP$ defines an operator in $\mathcal{P}(X, W)$ and $T = \overline{T}_W$. From this and according to Remark 4.3, if one of the conditions (i), (ii), (iii) or (iv) is valid, then $p(T) = p(\overline{T}_W) = \infty$ or $q(T) = q(\overline{T}_W) = \infty$. But, by Lemma 3.3, $p(\overline{T}) = \infty$ or $q(\overline{T}) = \infty$. Therefore by Theorem 4.1, we obtain the equalities (i)–(iv). $\square$

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