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NEW GENERALIZED APOSTOL-FROBENIUS-EULER POLYNOMIALS AND THEIR MATRIX APPROACH

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Abstract. In this paper, we introduce a new extension of the generalized Apostol-Frobenius-Euler polynomials $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$. We give some algebraic and differential properties, as well as, relationships between this polynomials class with other polynomials and numbers. We also, introduce the generalized Apostol-Frobenius-Euler polynomials matrix $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$ and the new generalized Apostol-Frobenius-Euler matrix $\mathcal{U}^{[m-1,\alpha]}(c,a;\lambda;u)$, we deduce a product formula for $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$ and provide some factorizations of the Apostol-Frobenius-Euler polynomial matrix $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$, which involving the generalized Pascal matrix.

1. Introduction

It is well-known that generalized Frobenius-Euler polynomial $H_n^{(\alpha)}(x;u)$ of order α is defined by means of the following generating function

$$\left(\frac{1-u}{e^z-u}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x;u)\frac{z^n}{n!}, \tag{1.1}$$

where $u \in \mathbb{C}$ and $\alpha \in \mathbb{Z}$. Observe that $H_n^{(1)}(x;u) = H_n(x;u)$ denotes the classical Frobenius-Euler polynomials and $H_n^{(\alpha)}(0;u) = H_n^{(\alpha)}(u)$ denotes the Frobenius-Euler numbers of order α. $H_n(x;-1) = E_n(x)$ denotes the Euler polynomials (see [2, 7]).

For parameters $\lambda, u \in \mathbb{C}$ and $a, b, c \in \mathbb{R}^+$, the Apostol type Frobenius-Euler polynomials $H_n(x;\lambda;u)$ and the generalized Apostol-type Frobenius-Euler polynomials are

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defined by means of the following generating functions (see [8]):

$$\left(\frac{1-u}{\lambda e^z-u}\right)e^{xz}=\sum_{n=0}^{\infty}H_n(x;\lambda;u)\frac{z^n}{n!}, \tag{1.2}$$

$$\left(\frac{a^z-u}{\lambda b^z-u}\right)^{\alpha}c^{xz}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(x;a,b,c;\lambda;u)\frac{z^n}{n!}. \tag{1.3}$$

If we set $x=0$ and $\alpha=1$ in (1.3), we get

$$\frac{a^z-u}{\lambda b^z-u}=\sum_{n=0}^{\infty}H_n(a,b,c;\lambda;u)\frac{z^n}{n!},$$

$H_n(a,b,c;u;\lambda)$ denotes the generalized Apostol-type Frobenius-Euler numbers (see [8]).

In the present paper, we introduce a new class of Frobenius-Euler polynomials considering the work of [8], we give relationships between this polynomials whit other polynomials and numbers, as well as the generalized Apostol-Frobenius-euler polynomials matrix.

The paper is organized as follows. Section 2 contains the definitions of Apostol-type Frobenius-Euler and generalized Apostol-Frobenius-Euler polynomials and some auxiliary results. In Section 3, we define the generalized Apostol-type Frobenius-Euler polynomials and prove some algebraic and differential properties of them, as well as their relation with the Stirling numbers of second kind. Finally, in Section 4 we introduce the generalized Apostol-type Frobenius-Euler polynomial matrix, derive a product formula for it and give some factorizations for such a matrix, which involve summation matrices and the generalized Pascal matrix of first kind in base c, respectively.

2. Previous Definitions and Notations

Throughout this paper, we use the following standard notions: $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\{0,1,2,\ldots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Furthermore, $(\lambda_0)=1$ and

$$(\lambda)_k=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1),$$

where $k\in\mathbb{N}$, $\lambda\in\mathbb{C}$. For the complex logarithm, we consider the principal branch. All matrices are in $M_{n+1}(\mathbb{K})$, the set of all $(n+1)\times(n+1)$ matrices over the field $\mathbb{K}$, with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Also, for i,j any nonnegative integers we adopt the following convention

$$\binom{i}{j}=0, \qquad \text{whenever } j>i.$$

Now, let us givel some properties of the generalized Apostol-type Frobenius-Euler polynomials and generalized Apostol-type Frobenius-Euler polynomials with parameters λ,a,c, order α (see [4,8,11]).

Proposition 2.1. *For a $m \in \mathbb{N}$, let $\{H_n^{(\alpha)}(x;u)\}_{n\geq 0}$ and $\{H_n(x;\lambda;u)\}_{n\geq 0}$ be the sequences of generalized Apostol-type Frobenius-Euler polynomials, generalized Frobenius-Euler polynomials respectively. Then the following statements hold.*

(a) *Special values: for $n \in \mathbb{N}_0$,*

$$H_n^{(0)}(x;u) = x^n.$$

(b) *Summation formulas:*

$$H_n^{(\alpha)}(x;u;a,b,c;\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(\alpha)}(x;u;a,b,c;\lambda)(x\ln c)^{n-k},$$

$$H_n^{(\alpha+\beta)}(x+y;u;a,b,c;\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(\alpha)}(x;u;a,b,c;\lambda)H_{n-k}^{(\beta)}(y;u;a,b,c;\lambda),$$

$$((x+y)\ln c)^n = H_{n-k}^{(\alpha)}(y;u;a,b,c;\lambda)H_k^{(-\alpha)}(x;u;a,b,c;\lambda),$$

$$H_n^{(-\alpha)}(x;u^2;a^2,b^2,c^2;\lambda^2) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(-\alpha)}(x;u;a,b,c;\lambda)H_{n-k}^{(-\alpha)}(x;-u;a,b,c;\lambda).$$

Definition 2.1. ([5, p. 207]). For $n \in \mathbb{N}_0$ and $x \in \mathbb{C}$, the Stirling numbers of second kind $S(n,k)$ are defined by means of the following expansion

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k).$$

The Jacobi polynomials of the degree n y orde (α,β), with $\alpha,\beta > -1$, the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be defined through Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left\{(1-x)^{n+\alpha}(1+x)^{n+\alpha}\right\}$$

and the values in the end points of the interval $[-1,1]$ is given by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \quad P_n^{(\alpha,\beta)}(-1) = (-1)^n\binom{n+\beta}{n}.$$

The relationship between the n-th monomial x^n and the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be written as

$$x^n = n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k}(-1)^k \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n+1}} P_k^{(\alpha,\beta)}(1-2x). \tag{2.1}$$

Proposition 2.2. *For $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\{B_n^{[m-1]}(x)\}_{n\geq 0}$, $\{G_n(x)\}_{n\geq 0}$ and $\{\mathcal{E}_n(x;\lambda)\}_{n\geq 0}$ be the sequences of generalized Bernoulli polynomials of level m, Genocchi polynomials and Apostol-Euler polynomials, respectively, we have the relationships:*

(a) [12, Equation (4)]

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x); \tag{2.2}$$

(b) [9, Remark 7]

$$x^n = \frac{1}{2(n+1)}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}G_k(x) + G_{n+1}(x)\right]; \tag{2.3}$$

(c) [10, Equation (32)]

$$x^n = \frac{1}{2}\left[\lambda\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(x;\lambda) + \mathcal{E}_n(x;\lambda)\right]. \tag{2.4}$$

Definition 2.2. Let x be any nonzero real number. For $c \in \mathbb{R}^+$, the generalized Pascal matrix of first kind in base c $P_c[x]$ is an $(n+1)\times(n+1)$ matrix whose entries are given by (see [13, 14])

$$p_{i,j,c}(x) := \begin{cases} \binom{i}{j}(x\ln c)^{i-j}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

When $c = e$, the matrix $P_c[x]$ coincides with the generalized Pascal matrix of first kind $P[x]$. Furthermore, if we adopt the convention $0^0 = 1$, then $P_c[0] = I_{n+1}$, with $I_{n+1} = \operatorname{diag}(1, 1, \ldots, 1)$.

An immediate consequence of the remarks above is the following proposition.

Proposition 2.3 (Addition Theorem of the argument)**.** *For $x, y \in \mathbb{R}$ is fulfilled*

$$P_c[x+y] = P_c[x]P_c[y].$$

Proposition 2.4. *For $c \in \mathbb{R}^+$, let $P_c[x]$ be the generalized Pascal matrix of first kind in base c and order $n+1$. Then the following statements hold.*

(a) *$P_c[x]$ is an invertible matrix and its inverse is given by*

$$P_c^{-1}[x] := (P_c[x])^{-1} = P_c[-x].$$

(e) *The matrix $P_c[x]$ can be factorized as follows*

$$P_c[x] = G_{n,c}[x]G_{n-1,c}[x]\cdots G_{1,c}[x], \tag{2.5}$$

where $G_{k,c}[x]$ is the $(n+1)\times(n+1)$ summation matrix given by

$$G_{k,c}[x] = \begin{cases} \begin{bmatrix} I_{n-k} & 0 \\ 0 & S_{k,c}[x] \end{bmatrix}, & k = 1, \ldots, n-1, \\ S_{n,c}[x], & k = n, \end{cases}$$

being $S_{k,c}[x]$ the $(k+1)\times(k+1)$ matrix whose entries $S_{k,c}(x; i, j)$ are given by

$$S_{k,c}(x; i, j, c) = \begin{cases} (x\ln c)^{i-j}, & i \geq j, \\ 0, & j > i, \end{cases} \qquad 0 \leq i, j \leq k.$$

3. Generalized Apostol-Frobenius-Euler Polynomials $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$

Definition 3.1. For $m\in\mathbb{N}$, $\alpha,\lambda,u\in\mathbb{C}$ and $a,c\in\mathbb{R}^+$, the generalized Apostol-type Frobenius-Euler polynomials in the variable x, parameters c,a,λ, order α and level m, are defined through the following generating function

$$\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{\alpha} c^{xz}=\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(x;c;a;\lambda;u)\frac{z^n}{n!}, \tag{3.1}$$

where $|z|<\left|\frac{\ln(u^m)}{\ln(c)}-\frac{\ln(\lambda)}{\ln(c)}\right|$.

For $x=0$ we obtain, the generalized Apostol-Frobennius-Euler numbers of parameters $\lambda\in\mathbb{C}$, $a,c\in\mathbb{R}^+$, order $\alpha\in\mathbb{C}$ and level $m\in\mathbb{N}$

$$\mathcal{H}_n^{[m-1,\alpha]}(c,a;\lambda;u):=\mathcal{H}_n^{[m-1,\alpha]}(0;c,a;\lambda;u).$$

According to the Definition 3.1, with $e=\exp(1)$, we have (1.1) and (1.2)

$$\mathcal{H}_n^{[0,\alpha]}(x;e,1;1;u)=H_n^{(\alpha)}(x;\lambda;u),$$
$$\mathcal{H}_n^{[0,1]}(x;e,1;\lambda;u)=H_n^{(1)}(x;\lambda;u).$$

Example 3.1. For any $\lambda\in\mathbb{C}$, $m=2$, $c=2$, $a=3$, $\alpha=\frac{1}{2}$ and $u=2$ the first the generalized Apostol-type Frobenius-Euler polynomials in the variable x, parameters c,a,λ, order α and level m are:

$$\mathcal{H}_0^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2)=\sqrt{\frac{3}{\lambda-4}},$$
$$\mathcal{H}_1^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2)=\sqrt{\frac{-3}{\lambda-4}}x\left[\frac{1}{2}\left(\frac{\ln 3}{\lambda-4}+\frac{3\lambda\ln 2}{(\lambda-4)^2}\right)+x\ln 4\right],$$
$$\begin{aligned}\mathcal{H}_2^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2)=\frac{1}{2}x^2\Bigg[&\left(\frac{-3}{4}\sqrt{\frac{-3}{\lambda-4}}\left(\frac{\ln 3}{\lambda-4}+\frac{3\lambda\ln 2}{(\lambda-4)^2}\right)^2\right.\\ &\left.+\frac{1}{2}\sqrt{\frac{-3}{\lambda-4}}\frac{-2\ln 3\ln 2}{(\lambda-4)^2}-\frac{6\lambda^2\ln 4}{(\lambda-4)^3}+\frac{3\lambda\ln 4}{(\lambda-4)^2}\right)\\ &+x\ln 2\sqrt{\frac{-3}{\lambda-4}}\left(\frac{\ln 3}{\lambda-4}+\frac{3\ln 2}{(\lambda-4)^4}\right)+x^2\ln 4\sqrt{\frac{-3}{\lambda-4}}\Bigg].\end{aligned}$$

Example 3.2. For any $\lambda\in\mathbb{C}$, $m=4$, $c=2$, $a=3$, $\alpha=1$ and $u=2$ the first the generalized Apostol-type Frobenius-Euler polynomials in the variable x, parameters c,a,λ, order α and level m are:

$$\mathcal{H}_0^{[3,1]}(x;2,3;\lambda;2)=\frac{-15}{\lambda-16},$$

$$\mathcal{H}_1^{[3,1]}(x;2,3;\lambda;2)=x\left[\frac{\ln 3}{\lambda-16}+\frac{\lambda 15\ln 2}{(\lambda-16)^2}-x\frac{15\ln 2}{\lambda-16}\right],$$

$$\mathcal{H}_2^{[3,1]}(x;2,3;\lambda;2)=\frac{1}{2}x^2\left[\frac{\ln 9}{\lambda-16}-\lambda\frac{2\ln 3\ln 2}{(\lambda-16)^2}+x\frac{2\ln 3\ln 2}{\lambda-16}-\lambda^2\frac{30\ln 4}{(\lambda-16)^3}\right.$$
$$\left.+x\frac{30\lambda\ln 4}{(\lambda-16)^2}+\lambda\frac{15\ln 4}{(\lambda-16)^2}-x^2\frac{15\ln 4}{\lambda-16}\right].$$

Example 3.3. For any $\lambda\in\mathbb{C}$, $m=2$, $c=3$, $a=e$, $\alpha=\frac{1}{3}$, and $u=5$ the first the generalized Apostol-type Frobenius-Euler polynomials in the variable x, parameters c,a,λ, order α and level m are:

$$\mathcal{H}_0^{[1,(\frac{1}{3})]}(x;3,e;\lambda;5)=\sqrt[3]{\frac{-24}{\lambda-25}},$$

$$\mathcal{H}_1^{[1,(\frac{1}{3})]}(x;3,e;\lambda;5)=x\left[\frac{1}{3}\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^2}\left(\frac{\omega}{\lambda-25}+\lambda\frac{24\ln 3}{(\lambda-25)^2}\right)\right.$$
$$\left.+x\ln 3\sqrt[3]{\frac{-24}{\lambda-25}}\right],$$

$$\mathcal{H}_2^{[1,(\frac{1}{3})]}(x;3,e;\lambda;5)=\frac{1}{2}x^2\left[\left(\frac{2}{9}\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^5}\frac{\omega}{\lambda-25}+\lambda\frac{24\ln 3}{(\lambda-25)^2}\right)^2\right.$$
$$+\frac{2}{3}x\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^2}\ln 3\left(\frac{\omega}{\lambda-25}+\lambda\frac{24\ln 3}{(\lambda-25)^2}\right)$$
$$+\frac{1}{3}\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^2}\left(-2\ln 3\frac{\omega}{(\lambda-25)}-\lambda^2\frac{-48\ln 9}{(\lambda-25)^3}\right.$$
$$\left.\left.+\lambda\frac{24\ln 9}{(\lambda-25)^2}\right)+x^2\ln 9\sqrt[3]{\frac{-24}{\lambda-25}}\right],$$

where $\omega=\ln\left(\frac{3060513257434037}{1125899906842624}\right)$.

Theorem 3.1. *For $m\in\mathbb{N}$, let $\{\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)\}_{n\geq 0}$ be the sequence of generalized Apostol-type Frobenius-Euler polynomials, whit parameters $\lambda,u\in\mathbb{C}$ and $a,c\in\mathbb{R}^+$, order $\alpha\in\mathbb{C}$ and level m. Then the following statements hold.*

(a) *For every $\alpha=0$ and $n\in\mathbb{N}_0$*
$$\mathcal{H}_n^{[m-1,0]}(x;c;a;\lambda;u)=(x\ln c)^n.$$

(b) *For $\alpha,\lambda\in\mathbb{C}$ and $n,k\in\mathbb{N}_0$, we have the relationship*
$$\mathcal{H}_n^{[m-1,\alpha]}(x;c;a;\lambda;u)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k}^{[m-1,\alpha]}(c;a;\lambda;u)(x\ln c)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha-1]}(c;a;\lambda;u)\mathcal{H}_{k}^{[m-1,1]}(x;c;a;\lambda;u).$$

(c) *Differential relations. For $m \in \mathbb{N}$ and $n, j \in \mathbb{N}_0$ with $0 \leq j \leq n$, we have*

$$[\mathcal{H}_{n}^{[m-1,\alpha]}(x;c;a;\lambda;u)]^{(j)} = \frac{n!}{(n-j)!}(\ln c)^{j}\, \mathcal{H}_{n-j}^{[m-1,\alpha]}(x;c,a;\lambda;u).$$

(d) *Integral formulas. For $m \in \mathbb{N}$, is fulfilled*

$$\int_{x_0}^{x_1} \mathcal{H}_{n}^{[m-1,\alpha]}(x;c,a;\lambda;u)\, dx = \frac{\ln c}{n+1}\left[\mathcal{H}_{n+1}^{[m-1,\alpha]}(x_1;c,a;\lambda;u) - \mathcal{H}_{n+1}^{[m-1,\alpha]}(x_0;c,a;\lambda;u)\right].$$

(e) *Addition theorem of the argument.*

$$\mathcal{H}_{n}^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\mathcal{H}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda;u), \tag{3.2}$$

$$\mathcal{H}_{n}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u)(x\ln c)^{k}, \tag{3.3}$$

$$((x+y)\log c)^{n} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c;a;\lambda;u)\mathcal{H}_{k}^{[m-1,-\alpha]}(x;c;a;\lambda;u). \tag{3.4}$$

Proof. (3.2) From Definition 3.1, we have

$$\sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha+\beta]}(x+y,c,a;\lambda;u)\frac{t^{n}}{n!}$$

$$= \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^{h}}{h!} - u^{m}}{\lambda c^{z} - u^{m}}\right]^{(\alpha+\beta)} c^{(x+y)z}$$

$$= \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^{h}}{h!} - u^{m}}{\lambda c^{z} - u^{m}}\right]^{\alpha} c^{xz} \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^{h}}{h!} - u^{m}}{\lambda c^{z} - u^{m}}\right]^{\beta} c^{yz}$$

$$= \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha]}(x;c;a;\lambda;u)\frac{z^{n}}{n!} \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\beta]}(y;c;a;\lambda;u)\frac{z^{n}}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{k}^{[m-1,\alpha]}(x,c,a;\lambda;u)\mathcal{H}_{n-k}^{[m-1,\beta]}(y,c,a;\lambda;u)\frac{z^{n}}{n!}.$$ □

Proof. (3.4) Making an adequate modification $\beta = -\alpha$ and aplply (3.2)

$$\sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha+\beta]}(x+y;c;a;\lambda;u)\frac{z^{n}}{n!}$$

$$=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{(\alpha+\beta)} c^{(x+y)z}$$

$$=\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{\alpha} c^{xz}\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{\beta} c^{yz}$$

$$=\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(x;c;a;\lambda;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,-\alpha]}(y;c;a;\lambda;u)\frac{z^n}{n!}$$

$$=c^{(x+y)z}$$

$$=\sum_{n=0}^{\infty}((x+y)\log c)^n\frac{z^n}{n!}.$$

Therefore, (3.4) holds. □

From (2.1) and Proposition 2.2 we deduce some algebraic relations connecting the polynomials $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$ with other families of polynomials.

Theorem 3.2. *For $m\in\mathbb{N}$, the generalized Apostol-type Frobenius-Euler polynomials of level m $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, by means of the identity.*

(3.5)

$$\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)$$
$$=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\alpha}{j-k}\binom{n}{j}\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{j+1}}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu))P_k^{(\alpha,\beta)}(1-2x).$$

Proof. By substituting (2.1) into the right-hand side of (3.3) and using appropriate binomial coefficient identities (see, for instance [1, 5, 6]), we see that

$$\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)$$
$$=\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)(n-j)!(\ln c)^{n-j}\sum_{k=0}^{n-j}(-1)^k\binom{n-j+\alpha}{n-j-k}$$
$$\times\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}}P_k^{(\alpha,\beta)}(1-2x)$$
$$=\sum_{j=0}^{n}\sum_{k=0}^{n-j}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)(n-j)!(\ln c)^{n-j}(-1)^k\binom{n-j+\alpha}{n-j-k}$$

$$\times \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}} P_k^{(\alpha,\beta)}(1-2x)$$
$$= \sum_{k=0}^{n}(-1)^k \sum_{j=0}^{n-k} \binom{n}{j}\binom{n-j+\alpha}{n-j-k} \mathcal{H}_j^{[m-1,\mu]}(y;c,a;\lambda;u)(n-j)!(\ln c)^{n-j}$$
$$\times \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}} P_k^{(\alpha,\beta)}(1-2x)$$
$$= \sum_{k=0}^{n}(-1)^k \sum_{j=k}^{n} j!(\ln c)^j \binom{j+\alpha}{j-k}\binom{n}{j} \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{j+1}}$$
$$\times \mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u) P_k^{(\alpha,\beta)}(1-2x).$$

Therefore, (3.5) holds. □

Theorem 3.3. *For $m \in \mathbb{N}$, the generalized Apostol-type Frobenius-Euler polynomials of level m $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the generalized Bernoulli polynomials of level m $B_n^{[m-1]}(x)$, by means of the following identity*

$$\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!(\ln c)^j}{(k+m)!}\binom{n}{j}\binom{j}{k} \mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu) B_{j-k}^{[m-1]}(x).$$

Proof. By substituting (2.2) into the right-hand side of (3.3), it suffices to follow the proof given in Theorem 3.2, making the corresponding modifications. □

Theorem 3.4. *For $m \in \mathbb{N}$, the generalized Apostol-type Frobenius-Euler polynomials of level m $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the Genocchi polynomials $G_n(x)$, by means of*

$$\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$$
$$=\frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\left[\binom{n}{k}\mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u) + \sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{j-k}\right] G_{k+1}(x). \tag{3.6}$$

Proof. By substituting (2.3) into the right-hand side of (3.3), we see that

$$\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$$
$$=\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)\frac{(\ln c)^{n-j}}{2(n-j+1)}\left[\sum_{k=0}^{n-j}\binom{n-j+1}{k+1}G_{k+1}(x) + G_{n-j+1}(x)\right]$$
$$=\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)\frac{(\ln c)^{n-j}}{2(n-j+1)}\sum_{k=0}^{n-j}\binom{n-j+1}{k+1}G_{k+1}(x)$$
$$+\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)\frac{(\ln c)^{n-j}}{2(n-j+1)}G_{n-j+1}(x).$$

Then, using appropriate combinational identities and summations (see, for instance [1, 5, 6]), we obtain

$$\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)$$

$$=\frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\left[\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{j-k}+\binom{n}{k}\mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u)\right]G_{k+1}(x).$$

Therefore, (3.6) holds. □

Theorem 3.5. *For $m\in\mathbb{N}$, the generalized Apostol-type Frobeniu-Euler polynomials of level m $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$, by means of the following identity*

$$\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)=\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\left[\lambda\mathcal{H}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;u)+(\ln c)^j\mathcal{H}_n^{[m-1,\alpha]}(y;c,a;\lambda;u)\right]\mathcal{E}_{n-j}(x;\lambda). \tag{3.7}$$

Proof. By substituting (2.4) into the right-hand side of (3.3), we can see that

$$\begin{aligned}
&\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)\\
&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{1}{2}\right)\left[\lambda\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)+\mathcal{E}_{n-k}(x;\lambda)\right]\\
&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)\\
&\quad+\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{1}{2}\right)\mathcal{E}_{n-k}(x;\lambda).
\end{aligned} \tag{3.8}$$

The first sum in (3.8) becomes

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)\\
&=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\binom{n-k}{j}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)\mathcal{E}_j(x;\lambda)\\
&=\sum_{j=0}^{n}\left(\frac{\lambda}{2}\right)\binom{n}{j}\mathcal{E}_j(x;\lambda)\sum_{k=0}^{n-j}\binom{n-j}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\\
&=\sum_{j=0}^{n}\left(\frac{\lambda}{2}\right)\binom{n}{j}\mathcal{E}_j(x;\lambda)\mathcal{H}_{n-j}^{[m-1,\alpha]}(y+1;c,a;\lambda;u).
\end{aligned} \tag{3.9}$$

For the second sum in (3.8), we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{1}{2}\right)\mathcal{E}_{n-k}(x;\lambda)\\
&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{k}\mathcal{E}_k(x;\lambda).
\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) we get

$$\begin{aligned}
&\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)\\
=&\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j(x;\lambda)\mathcal{H}_{n-j}^{[m-1,\alpha]}(y+1;c,a;\lambda;u)\\
&+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^j\mathcal{E}_j(x;\lambda)\\
=&\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\left[\lambda\mathcal{H}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;u)+(\ln c)^j\mathcal{H}_n^{[m-1,\alpha]}(y;c,a;\lambda;u)\right]\mathcal{E}_{n-j}(x;\lambda).
\end{aligned}$$

Therefore, (3.7) holds. □

Proposition 3.1. *For $m\in\mathbb{N}$, $\alpha,\lambda,u,\in\mathbb{C}$, $a,c\in\mathbb{R}^+$ and $n\in\mathbb{N}_0$, we have*

$$\begin{aligned}
\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)&=\sum_{k=0}^{n}k!\binom{x}{k}\sum_{j=0}^{n-k}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-j}S(n-j,k)\\
&=\sum_{k=0}^{n}k!\binom{x}{k}\sum_{j=k}^{n}\binom{n}{n-j}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{j}S(j,k).
\end{aligned}$$

4. The Generalized Apostol-Frobenius-Euler Polynomials Matrix

Definition 4.1. The generalized $(n+1)\times(n+1)$ Apostol-Frobenius-Euler polynomials matrix $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$ with $m\in\mathbb{N}$, $\alpha,\lambda,u\in\mathbb{C}$ and a,c positive real numbers is defined by

$$\mathcal{U}_{i,j}^{[m-1,\alpha]}(x;c,a;\lambda;u)=\begin{cases}\binom{i}{j}\mathcal{H}_{i-j}^{[m-1,\alpha]}(x;c,a;\lambda;u), & i\geq j,\\ \\ 0, & \text{otherwise.}\end{cases}$$

While, the matrices

$$\begin{aligned}
\mathcal{U}^{[m-1]}(x;c,a;\lambda;u)&:=\mathcal{U}^{[m-1,1]}(x;c,a;\lambda;u),\\
\mathcal{U}^{[m-1]}(c,a;\lambda;u)&:=\mathcal{U}^{[m-1]}(0;c,a;\lambda;u)
\end{aligned}$$

are called the Apostol-Frobenius-Euler polynomial matrix and the Apostol-Frobenius-Euler matrix, respectively.

Since $\mathcal{H}_n^{[m-1,0]}(x;c,a;\lambda;u)=(x\ln(c))^n$, we have $\mathcal{U}^{[m-1,0]}(x;c,a;\lambda;u)=P_c[x]$. It is clear that (3.3) yields the following matrix identity:

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u)=\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)P_c[x].$$

Theorem 4.1. *For a fixed $m\in\mathbb{N}$, let $\{\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)\}_{n\geq0}$ and $\{\mathcal{H}_n^{[m-1,\beta]}(x;c,a;\lambda;u)\}_{n\geq0}$ be the sequences of generalized Apostol-type Frobenius-Euler* *polynomials in the variable x, parameters* $\lambda, u \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, *order* $\alpha \in \mathbb{C}$ *and level* m. *Then satisfies the following product formula:*

$$\begin{aligned}\mathcal{U}^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;u) &= \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\beta]}(x;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\beta]}(x;c,a;\lambda;u).\end{aligned} \tag{4.1}$$

Proof. Let $B_{i,j,c}^{[m-1,\alpha,\beta]}(a;\lambda;u)(x,y)$ be the (i,j)-th entry of the matrix product $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\beta]}(y;c,a;\lambda;u)$, then by the addition formula (3.2) we have

$$\begin{aligned}B_{i,j,c}^{[m-1,\alpha,\beta]}(a;\lambda;u)(x,y) &= \sum_{k=0}^{n}\binom{i}{k}\mathcal{H}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\binom{k}{j}\mathcal{H}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \sum_{k=j}^{i}\binom{i}{k}\mathcal{H}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\binom{k}{j}\mathcal{H}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}\mathcal{H}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\mathcal{H}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\mathcal{H}_{i-j-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\mathcal{H}_{k}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \binom{i}{j}\mathcal{H}_{i-j}^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;u),\end{aligned}$$

which implies the first equality of the theorem. The second and third equalities of can be derived in a similar way. ☐

Corollary 4.1. *For a fixed* $m \in \mathbb{N}$, *let* $\{\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)\}_{n\geq 0}$ *and* $\{\mathcal{H}_n^{[m-1,\beta]}(x;c,a;\lambda;u)\}_{n\geq 0}$ *be the sequences of generalized Apostol-type Frobenius-Euler polynomials in the variable x, parameters* $\lambda, u \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, *order* $\alpha \in \mathbb{C}$ *and level* m *and* $P_c[x]$ *the generalized Pascal matrix of first kind in base c. Then*

$$\begin{aligned}\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) &= \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)P_c[y]\\ &= P_c[x]\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)P_c[x].\end{aligned}$$

In particular,

$$\begin{aligned}\mathcal{U}^{[m-1]}(x+y;c,a;\lambda;u) &= P_c[x]\mathcal{U}^{[m-1]}(y;c,a;\lambda;u)\\ &= P_c[y]\mathcal{U}^{[m-1]}(x;c,a;\lambda;u).\end{aligned}$$

Proof. The substitution $\beta = 0$ into (4.1) yields

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)\mathcal{U}^{[m-1,0]}(y;c,a;\lambda;u).$$

Since $\mathcal{U}^{[m-1,0]}(y;c,a;\lambda;u) = P_c[y]$, we obtain

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)P_c[y]. \tag{4.2}$$

A similar argument allows to show that

$$\begin{aligned}\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) &= P_c[x]\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)P_c[x].\end{aligned}$$

Finally, the substitution $\alpha = 1$ into (4.2) and its combination with the previous equations completes the proof. □

Using the relation (2.5) and Corollary 4.1 we obtain the following factorization for $\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u)$ in terms of summation matrices.

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)G_{n,c}[y]G_{n-1,c}[y]\cdots G_{1,c}[y].$$

Under the appropriate choice on the parameters, level and order, it is possible to provide some illustrative examples of the generalized Apostol-Frobenius-Euler polynomials matrices.

Example 4.1. For $m = 1$, $c = a = e = \exp(1)$, $\alpha = 1$, $\lambda = -1$, The first four polynomials $\mathcal{H}_k^{[1-1,1]}(x;e,e;1;u)$, $k = 0,1,2,3$ are

$$\begin{aligned}\mathcal{H}_0^{[1-1,1]}(x;e,e;1;u) &= 1,\\ \mathcal{H}_1^{[1-1,1]}(x;e,e;1;u) &= x - \frac{1}{1-u},\\ \mathcal{H}_2^{[1-1,1]}(x;e,e;1;u) &= x^2 - \frac{2}{1-u}x + \frac{1+u}{(1-u)^2},\\ \mathcal{H}_3^{[1-1,1]}(x;e,e;1;u) &= x^3 - \frac{3}{1-u}x^2 + \frac{3(1+u)}{(1-u)^2}x - \frac{u^2+4u+1}{(1-u)^3}.\end{aligned}$$

Hence, for $n = 3$, we have

$$\mathcal{U}^{[m-1,1]}(x;e,e;1;u) = \begin{bmatrix} 1 & 0 & 0 & 0\\ u_{10} & 1 & 0 & 0\\ u_{20} & u_{21} & 1 & 0\\ u_{30} & u_{31} & u_{32} & 1\end{bmatrix},$$

where

$$\begin{aligned}u_{10} = u_{21} = u_{32} &= \mathcal{H}_1^{[1-1,1]}(x;e,e;1;u),\\ u_{20} = u_{31} &= \mathcal{H}_2^{[1-1,1]}(x;e,e;1;u),\\ u_{30} &= \mathcal{H}_3^{[1-1,1]}(x;e,e;1;u).\end{aligned}$$

Example 4.2. For $m = 1$, $c = a = e = \exp(1)$, $\lambda = 1$ and $u = -1$, The first four polynomials $\mathcal{H}_k^{[1-1,\alpha]}(x; e, e; 1; -1)$, $k = 0, 1, 2, 3$, are

$$\begin{aligned}
\mathcal{H}_0^{[1-1,\alpha]}(x; e, e; 1; -1) &= 1, \\
\mathcal{H}_1^{[1-1,\alpha]}(x; e, e; 1; -1) &= x - \frac{\alpha}{2}, \\
\mathcal{H}_2^{[1-1,\alpha]}(x; e, e; 1; -1) &= x^2 - \alpha x + \frac{\alpha(\alpha-1)}{4}, \\
\mathcal{H}_3^{[1-1,\alpha]}(x; e, e; 1; -1) &= x^3 - \frac{3\alpha}{2}x^2 + \frac{3\alpha(\alpha-1)}{4}x - \frac{3\alpha^2(\alpha-1)}{8}.
\end{aligned}$$

Then, for $n = 3$, we have

$$\mathcal{U}^{[m-1,\alpha]}(x; e, e; 1; -1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u_{10} & 1 & 0 & 0 \\ u_{20} & 2u_{21} & 1 & 0 \\ u_{30} & 3u_{31} & 3u_{32} & 1 \end{bmatrix},$$

where

$$\begin{aligned}
u_{10} = u_{21} = u_{32} &= \mathcal{H}_1^{[1-1,\alpha]}(x; e, e; 1; -1), \\
u_{20} = u_{31} &= \mathcal{H}_2^{[1-1,\alpha]}(x; e, e; 1; -1), \\
u_{30} &= \mathcal{H}_3^{[1-1,\alpha]}(x; e, e; 1; -1).
\end{aligned}$$

Example 4.3. For $\lambda \in \mathbb{C}$, $m = c = 2$, $a = 3$, $\alpha = \frac{1}{2}$, $u = 2$, we have the Example 3.1. Therefore,

$$\mathcal{U}^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) \quad = \quad \begin{bmatrix} \sqrt{\frac{3}{\lambda-4}} & 0 & 0 \\ \mathcal{H}_1^{[1,(\frac{1}{2})]}(x; 2, 3; \lambda; 2) & \sqrt{\frac{3}{\lambda-4}} & 0 \\ \frac{32}{\sqrt{1+\lambda}} & 0 & 0 \\ \mathcal{H}_2^{[1,(\frac{1}{2})]}(x; 2, 3; \lambda; 2) & 2\mathcal{H}_1^{[1,(\frac{1}{2})]}(x; 2, 3; \lambda; 2) & \sqrt{\frac{3}{\lambda-4}} \end{bmatrix}.$$

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