Advances in Nonlinear Analysis on Mechanical Systems - Research Article

Advances in Mechanical Engineering

Advances in Mechanical Engineering
2021, Vol. 13(8) 1–9

DOI: 10.1177/16878140211039138
journals.sagepub.com/home/ade
SAGE

Formulation and existence of weak solutions for a problem of adhesive contact with elastoplasticity and hardening

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Abstract
This paper presents the weak formulation of a quasi-static evolution model for two deformable bodies with unidirectional adhesive unilateral contact on which external loads act. Small deformations and linearized elastoplasticity with hardening are assumed. The adhesion component is rate-dependent or rate-independent according to the choice of the viscosity coefficient of the glue; elastoplasticity is considered rate-independent. The weak formulation is expressed as a doubly non-linear problem with unbounded multivalued operators, as a function of internal and boundary displacements, plastic and symmetric strain tensors, and the bonding field and its gradient. This paper differs from other formulations by coupling the equations defined inside and on the boundary of the solids in functional form. In addition to this novelty, we verify the existence of solutions by a path other than that displayed in similar articles. The existence of solutions is demonstrated after considering a succession of doubly non-linear problems with an unbounded operator, and verifying that the solution of one of the problems is also a solution to the objective problem. The proof is supported by previous results from non-linear Partial differential equations theory with monotone operators.

Keywords
Contact, delamination, unilateral, unidirectional, doubly non-linear problem, strongly monotone operator

Date received: 5 January 2021; accepted: 16 July 2021

Handling editor: James Baldwin

Introduction

Quasi-static problems of adhesive contact with elastoplasticity have focused on energetic solutions, for example, the case of contact and elastoplasticity with hardening,[1] delamination problems,[2,3] the rate-independent model with damage,[4] adhesive contact with temperature,[5] the numerical approach developed in Panagiotopoulos et al.,[6] among others. A novelty of this document consists of the weak formulation from standard models of elastoplasticity and adhesion (which considers interactions between the bonding field and the displacement on the border), together with the proof of the existence of weak solutions (without going into the field of the energetic solutions). We also obtained an abstract representation of the set of equations in a single differential inclusion with unbounded multivalued operators. The problem is described below.

Two deformable solids occupying reference domains Ω_1, Ω_2 are considered in adhesive contact in a common region $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ (see Figure 1). Both solids with a

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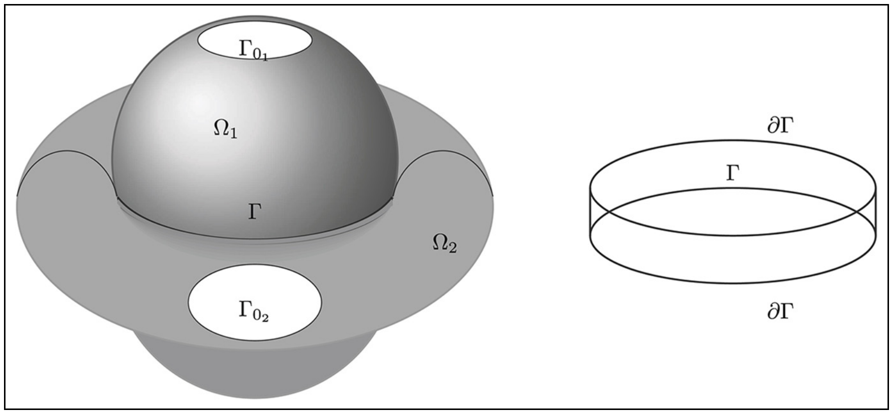

Figure 1. Ω_1 adhered to Ω_2 in Γ, with prescribed displacement in $\Gamma_{0_1}, \Gamma_{0_2}$. A figure made with Latex Tikz package.

boundary $\Gamma_{0_i} \subsetneqq \partial\Omega_i \backslash \Gamma$, with prescribed displacement ($u^i_{|\Gamma_{0_i}} = 0, i = 1, 2$). For each material, we consider a dissipation potential $R : U \to]-\infty, +\infty]$ and a stored energy density $W: U \to]-\infty, +\infty[$, where U is a Hilbert space for (e, p) (e is the linearized symmetric strain tensor and p is the plastic tensor). The elastoplastic component is defined by a momentum equation and a plastic flow rule as a function of W and R. The treatment in this paper does not include damage and temperature. We consider a unidirectional (irreversible) and unilateral (no penetration between solids) adhesive contact through differential Fremond inclusions on the contact boundary.[7] The bonding field β, the displacements on the boundary $u_{|\Gamma}$ and indicator functions define these differential inclusions. There is extensive literature on adhesive contact problems in Refs.[8–10] among other works.

Doubly non-linear problems have been mainly addressed by Refs.[11–14] The weak model will have the abstract representation $\partial\varphi(\dot{z}) + \partial\psi(z) \ni F$, where $\partial\varphi$ and $\partial\psi$ are unbounded multivalued monotone operators. The proof of the existence of weak solutions cannot be direct since the operators $\partial\varphi$ and $\partial\psi$ are not bounded. For this reason, we will construct a sequence of doubly non-linear problems that present solutions, and we will prove that one of them also solves the original problem. The contributions to be considered in this work are: the strong monotony proof of $\partial\psi$ (Theorem 11), the proof of the existence of solutions (Theorem 14), and the weak formulation based on its extended variables and the loads applied to the interior and exterior of solids (Theorem 6).

The document has been organized as follows: Section 2 describes the classical formulation of elastoplastic models and differential inclusions for adhesion. It also defines the monotone operators, the doubly nonlinear problems, and presents a theorem of existence and uniqueness of solutions. Section 3 defines the weak formulation of the problem, as well as the solution spaces. Finally, Section 4 demonstrates the existence of solutions of the weak model. The reader who is not very interested in the mathematical foundations of the model can omit the Sections 2.3 and 4 of this document.

Model equations

This section summarizes the basic concepts of elastoplasticity and adhesion and displays the constitutive equations of the model. The elastoplastic model with hardening that we present follows the variational formulation proposed by Stefanelli[15]; the notations and other considerations of the elastoplastic model as a rate-independent system are taken from Liero and Mielke.[16]

Linearized elastoplasticity as a rate-independent system

Rate-independent systems (RIS[17]) arise in various phenomena such as elastoplasticity, delamination damage, fracture propagation, ferroelectricity, among others. In a RIS problem, any monotonous reparameterization of the time variable in the solution of the problem solves the original problem with the respective reparameterization. Therefore, it does not display an intrinsic time scale.

Consider two domains $\Omega_i \subset \mathbb{R}^3$ $(i = 1, 2)$ with boundary $\partial\Omega_i$ of class C^1, and where $\Gamma_{0_i} \subset \partial\Omega_\mathrm{i}$ is the part of the boundary with the Dirichlet condition

$$\left. \begin{array}{l} u^1 = 0 \,\mathrm{on}\, \Gamma_{0_1} \\ u^2 = 0 \,\mathrm{on}\, \Gamma_{0_2} \end{array} \right\}, \quad (1)$$

and let $H^1(\Omega_i) := \left\{ u^i \in H^1(\Omega_i)^3 \ : \ u^i|_{\Gamma_{0_i}} = 0 \right\}$. It is also assumed that $meas(\Gamma_{0_i}) > 0, i = 1, 2$, so the pair $(\Omega_\mathrm{i}, \Gamma_{0_i})$ satisfies the Korn's inequality $\|e(u^i)\|_{L^2(\Omega_i)} \geqslant$

$C_K\|u^i\|_{H^1(\Omega_i)}$ for some $C_K>0$ and all $u^i \in H^1(\Omega_i)$, $i = 1, 2$.[18] Here, $e(u) = \frac{1}{2}(\nabla u + \nabla^\top u)$ is the symmetric tensor of linearized strain.

The elastoplastic properties of each body Ω_i are prescribed by the stored energy density $W_i(e,p)$, and the potential of dissipation $R_i(e)$, where:

- $e \in L^2(\Omega_i)^{3\times 3}_{sym} := L^2(\Omega_i; \mathbb{R}^{3\times 3}_{sym})$,
- $p \in L^2(\Omega_i)^{3\times 3}_{dev} := L^2(\Omega_i; \mathbb{R}^{3\times 3}_{dev})$,
- $\mathbb{R}^{3\times 3}_{dev} = \left\{A \in \mathbb{R}^{3\times 3}_{sym} \ : tr(A) = 0\right\}$,
- $W_i \ : \ L^2(\Omega_i)^{3\times 3}_{sym} \times L^2(\Omega_i)^{3\times 3}_{dev} \to \mathbb{R}$ is continuous and coercive,
- $R_i \ : \ L^2(\Omega_i; \mathbb{R}^{3\times 3}_{dev}) \to \mathbb{R}$ is continuous, convex and 1-homogeneous (i.e. $R(\lambda p) = \lambda R(p)$ for all $\lambda>0$, which guarantees that the material response is rate-independent).

The solution of a rate-independent linearized elastoplastic problem has to solve the *momentum equation*

$$-\ div(\partial_e W_i(e(u^i), p^i)) = f^i \ \ in \ \ \Omega_i, \tag{2}$$

and a differential inclusion called plastic flow rule

$$0 \in \partial R_i(\dot{p}\, i) + \partial_p W_i(e(u^i), p^i) \ \ in \ \ \Omega_i, \tag{3}$$

where $\partial R_i(p)$ is the subdifferential of R_i in p, and f_i it is a density of force per unit volume. Particularly we consider stored energy density to the quadratic functional (in the isotropic and homogeneous case)

$$\begin{aligned} W_i(e,p) &= \left\langle \frac{1}{2}C(e-p), (e-p) \right\rangle + \frac{h_i}{2}\|p\|^2 \\ &= \frac{\lambda_i}{2}(tre)^2 + \mu_i\|e-p\|^2 + \frac{h_i}{2}\|p\|^2, \end{aligned} \tag{4}$$

$i = 1, 2$, where $C(e) = \frac{\lambda_i}{2}(tre)I_d + 2\mu e$, λ_i, μ_i are the coefficients of Lamé and h_i a measure for the kinematic hardening. In this particular case, the stress tensor is determined by the derivative of the energy functional concerning the deformation tensor: $\sigma = \partial_e W\ (e(u), p)$. On the other hand, dissipation potential is considered as

$$R_i(p) = \sigma_{yield(\Omega_i)}\|p\|,$$

where σ_{yield} is the yield stress. According to (4), we have to $\sigma = \lambda(tre)I_d + 2\mu(e-p) \in \mathbb{R}^{3\times 3}_{sym}$, and $\partial_p W(e(u), p) = -2\mu(e-p) + hp$. On the system (2)–(3), we will assign the boundary conditions

$$\left.\begin{aligned} \sigma^1 N_1 &= \partial_e W_1(e(u^1), p^1)N_1 = g^1, \text{ on } \partial\Omega_1 \backslash \Gamma \\ \sigma^2 N_2 &= \partial_e W_2(e(u^2), p^2)N_2 = g^2, \text{ on } \partial\Omega_2 \backslash \Gamma \end{aligned}\right\} \tag{5}$$

where N_i denotes the normal external vector to $\partial\Omega_i$, g^i is a force density per unit area and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$.

Adhesion

A contact problem with small deformations is a system of constitutive equations or abstract equations, which models the deformation of two or more bodies under load. They can include effects such as damage, adhesion, memory, friction, temperature, and other dissipative responses. The classic adhesive contact models such as the one discussed in this paper are idealized cases that assume cohesive zones independent of time, and that generally introduce inconsistencies in the model (see Heitbreder et al.[19] for a more in-depth discussion); However, this analysis is outside the theoretical scope of this paper as is the thermodynamics associated with the boundary,[20] and instead we will consider the standard approach of Fremond[7] on adhesive contact.

Suppose that Ω_1 and Ω_2 are glued in a common region of contact Γ, it is assumed that Γ is of class C^1 and that it also satisfies the Korn inequality. In addition to the variable e, p introduced in Section 2.1, the variable $\beta : \Gamma \to [0, 1]$ models the evolution of the surface fraction with active glue fibers, which break or mend by microscopic motions (solid glue is assumed and for such irreversibility of the break). When $\beta = 0$ there are no active glue bonds, when $\beta = 1$ all bonds are active, when $0<\beta<1$, there is a proportion of active glue bonds.

Because of the conditions imposed on the border of solids, trace theorem extends the displacement field $u(x)$ with $x \in \Omega_i$ to each of the points $x \in \partial\Omega_i$, in particular, can be extended over Γ. It will be denoted by $\left\|u^2_{|_\Gamma} - u^1_{|_\Gamma}\right\|$ the gap on the contact surface Γ, where $u^i_{|\Gamma}$ is the small displacement of the solids Ω_i on Γ at the macroscopic level. For the sake of simplicity, it neglects the thermal phenomena and excludes the temperature as a state quantity. Also, no work involving microscopic motions is provided to the system, that is, there are no chemical, radiative, optical, or electrical actions. The differential inclusions used by Fremond for the adhesion problem are

$$\begin{aligned} c_s\dot{\beta} - k_s\Delta_s\beta + \partial I_{[0,1]}(\beta) + \partial I_-\left(\dot{\beta}\right) \\ \ni \omega_s - \frac{k}{2}\left\|u^2_{|_\Gamma} - u^1_{|_\Gamma}\right\|^2 \text{ on } \Gamma, \end{aligned} \tag{6}$$

$$\left.\begin{aligned} \sigma^1 N_1 - k\beta(u^2_{|_\Gamma} - u^1_{|_\Gamma}) - \partial I_-((u^2_{|_\Gamma} - u^1_{|_\Gamma}) \cdot N_2)N_2 \ni 0, \\ \sigma^2 N_2 + k\beta(u^2_{|_\Gamma} - u^1_{|_\Gamma}) + \partial I_-((u^2_{|_\Gamma} - u^1_{|_\Gamma}) \cdot N_2)N_2 \ni 0, \end{aligned}\right\} \text{ on } \Gamma, \tag{7}$$

where:

- the parameter k_s measures the intensity of microscopic interactions,
- k is an elastic constant of the adhesive material,
- c_s is the viscosity coefficient of the glue,

- $I_A(x) = \begin{cases} 0 \ if \ x \in A \\ +\infty \ if \ x \notin A \end{cases}$ is the indicator function ,
- $I_-(x) = \begin{cases} 0 \ if \ x \leq 0 \\ +\infty \ if \ x > 0 \end{cases}, \quad I_{[0,1]}(x) = \begin{cases} 0 \ if \ 1 \leq x \leq 0 \\ +\infty \ \text{otherwise} \end{cases},$
- $I_-((u^2_{|_\Gamma} - u^1_{|_\Gamma}) \cdot N_2)$ is an impenetrability constraint of both solids (unilateral contact),
- $I_-(\dot{\beta})$ characterizes the irreversible behavior of solid glues (uni-directional contact),
- ω_s is the energy of Dupré which consists of the work required to separate two adhered bodies.

In case of considering reversible adhesion, the relation $\partial I_-(\dot{\beta})$ is removed from the equation. The boundary and initial conditions for β are

$$k \frac{\partial \beta}{\partial n_s} = 0 \text{ on } \partial\Gamma, \quad (8)$$

$$\beta(x, 0) = \beta_0(x) \text{ on } \Gamma, \quad (9)$$

where n_s denoting the normal vector exterior to $\partial\Gamma$.

Doubly nonlinear problems

Several authors have studied doubly non-linear problems, including,[11,12,21] and the references in them. The differential inclusion related to this model has the representation

$$\partial\varphi(\dot{z}) + \partial\psi(z) \ni F. \quad (10)$$

The approach to be used corresponds to the developments made by Colli,[12] where $\partial\varphi$ and $\partial\psi$ are monotonous operators. The theorem that guarantees the existence of solutions of (10) is enunciated at the end of this section. Before that, the section presents some definitions and results of the convex analysis. In particular, we will use functions $f : \mathbf{V} \to]-\infty, +\infty]$ that are convex, lower semi-continuous (i.e. f is l.s.c. if $f(x) \leq \liminf_{y \to x} f(y)$ for every $x, y \in \mathbf{V}$), proper (i.e. $f(x) \neq \infty$ for some $x \in \mathbf{V}$), and sub-differentiable ($x^* \in \partial f(x)$ if $x^* \in V^*$ and $\langle x^*, y - x \rangle \leq f(y) - f(x)$ for every $y \in \mathbf{V}$) in $x \in \mathbf{V}$.

Example 1.

(1) *If $K \subset \mathbf{V}, K$ closed and convex, the indicator function I_K is lower semi-continuous and convex* (see Temam,[22] Prop 2.3).

(2) *If f is convex, proper and semi continuous in* $\mathbf{V}$, *then* $\partial f(x) \neq \emptyset$ (see Temam,[22] Cor 2.5, Prop 5.2).

The solution space is defined as a Cartesian product of spaces $L^2(\Omega)^{m \times n}$ and Sobolev spaces $W^{1,2}(\Omega)^{m \times n}$. Some distinctive results of these spaces are:

- There is a continuous immersion of $W^{1,2}(\Omega)$ in $L^2(\Omega)$,
- $W^{1,2}(\Omega)$ is a dense subspace of $L^2(\Omega)$,
- The containment of $W^{1,2}(\Omega)$ in $L^2(\Omega)$ is compact (usually symbolized by $W^{1,2}(\Omega) \Subset L^2(\Omega)$) (see e.g. Adams and Fournier,[23] Th 6.2 or Brezis,[24] Th 9.16).

Multivalued monotone operators are now introduced.

Definition 2. *Let* $\mathbf{V}$, $\mathbf{W}$ *linear spaces. A* multivalued operator *A is a relation from* $\mathbf{V}$ *to* $\mathbf{W}$ *($A : \mathbf{V} \rightrightarrows \mathbf{W}$), that is, $A : \mathbf{V} \to 2^{\mathbf{W}}$. In this case, $y \in A(x)$ if $(x, y) \in A$.*

Definition 3. *Let* $\mathbf{V}$ *be a Banach space and $A : \mathbf{V} \rightrightarrows \mathbf{V}^*$ a multivalued operator.*

(1) *A is* monotone *if for all (x_1, y_1), $(x_2, y_2) \in A$, it is verified*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0.$$

(2) *A is* strongly monotone *if there exists $C > 0$ such that for any (x_1, y_1), $(x_2, y_2) \in A$, one has*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq C \|x_1 - x_2\|^2_V.$$

It is time to consider the problem

$$A\left(\dot{z}(t)\right) + B(z(t)) \ni f(t) \quad \text{for a.e. } t \in]0, T[, z(0) \in \mathbf{V},$$

where $A : \mathbf{W} \rightrightarrows \mathbf{W}^*$, $B : \mathbf{V} \rightrightarrows \mathbf{V}^*$, $\mathbf{V} \subset \mathbf{W}$, and $f :]0, T[\to \mathbf{W}^*$. $\mathbf{W}$ is assumed to be reflexive and strictly convex ($\mathbf{V}$ is strictly convex if the sphere in $\mathbf{V}$ does not contain any line segment).

Theorem 4.

Let $\mathbf{V}, \mathbf{W}$ such that, $\mathbf{V} \Subset \mathbf{W}$, $\mathbf{V}$ dense in $\mathbf{W}$, and let φ, $\psi : \mathbf{W} \to]-\infty, +\infty]$ proper, convex, and l.s.c. functions such that

(1) *$\partial\varphi : \mathbf{W} \rightrightarrows \mathbf{W}^*$ is bounded (i.e. maps bounded sets into bounded sets),*

(2) *$\partial\psi : \mathbf{W} \rightrightarrows \mathbf{W}^*$ is strongly monotone in* $\mathbf{V}$,

(3) *$f \in L^1(0, T; \mathbf{W}^*) \cap H^1(0, T; \mathbf{V}^*)$,*

(4) *$z_0 \in \mathbf{V}$ and there exists $v_0 \in \partial\psi(z_0) \subset \mathbf{V}^*$,*

(5) *$f(0) - v_0 \in D(\varphi^*)$, where φ^* is the conjugate function ($\varphi^*(x^*) = \sup_{x \in \mathbf{V}}\{\langle x^*, x \rangle - \varphi(x)\}, x^* \in \mathbf{V}^*$) of φ,*

Then there exists a triple $z \in H^1(0, T; \mathbf{V})$, $w \in L^\infty(0, T; \mathbf{W}^*)$, $v \in L^1(0, T; \mathbf{W}^*) \cap L^\infty(0, T; \mathbf{V})$ satisfing

$$w(t) + v(t) = f(t),$$

$$w(t) \in \partial\varphi(\dot{z}(t)),$$

$$v(t) \in \partial\psi(z(t)) \ \text{ for a.e. } t \in]0, T[z(0) = z_0$$

Weak formulation

A variational or weak formulation of a physical model consists of rewriting the differential equations as functional acting on a space of functions. Such functionals defined by integrals will have a representation by derivatives whose order may decrease or even not exist by way of integration by parts. Due to the multiple connotations of the term "variational" it has been decided to call this section "weak formulation" so as not to confuse it with other formulations.[25] In this regard, this section states the weak model of adhesive and elastoplastic contact as a function of u, p, β and of the surface and volume loads acting on each solid in theorem 6.

We define the space **W** as the set of triples

$$(\mathrm{u}, \mathrm{p}, \beta)_W = \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \\ \beta \end{pmatrix}$$

such that $\mathrm{u} = (u^1, u^2) \in L^2(\Omega_1) \times L^2(\Omega_2)$, $\mathrm{p} = (p^1, p^2) \in L^2(\Omega_1)^{3\times 3} \times L^2(\Omega_2)^{3\times 3}$, $\beta \in L^2(\Gamma)$. Since Ω_1 and Ω_2 have boundaries of class C^1, the elements of $H^1(\Omega_1)^3 \times H^1(\Omega_2)^3$ can be extended to the boundary from the trace operator γ_0.[26] The representation of this extension will be the pair $(\mathrm{u}_{|\partial\Omega\backslash\Gamma}, \mathrm{u}_{|\Gamma})$, where $\mathrm{u}_{|\partial\Omega\backslash\Gamma} := (\gamma_0 \mathrm{u})_{|\partial\Omega\backslash\Gamma}, \mathrm{u}_{|\Gamma} := (\gamma_0 \mathrm{u})_{|\Gamma}$. We define

$$H^1 := \left\{ \mathrm{u} \in H^1(\Omega_1)^3 \times H^1(\Omega_2)^3 \ : \nabla u^i = \left(\nabla u^i\right)^\top, i = 1, 2 \right\},$$

$$H^{1/2} := \left\{ \begin{array}{ll} (\mathrm{u}_{|\partial\Omega\backslash\Gamma}, \mathrm{u}_{|\Gamma}) : & \mathrm{u}_{|\Gamma} = \mathrm{u}_{|\partial\Omega\backslash\Gamma} \mathrm{in} \partial\Gamma, u^i_{|\partial\Omega\backslash\Gamma} \in L^2(\partial\Omega_i\backslash\Gamma)^3, \\ & u^i_{|\Gamma} \in L^2(\Gamma)^3, i = 1, 2 \end{array} \right\},$$

and we represent with **V** the space of the vector functions

$$(\mathrm{u}, \mathrm{p}, \beta)_V := \begin{pmatrix} \mathrm{u} & \mathrm{e(u)} & \mathrm{u}_{|\partial\Omega\backslash\Gamma} & \mathrm{u}_{|\Gamma} \\ & & \mathrm{p} & \\ & \beta & \nabla\beta & \end{pmatrix},$$

such that $\mathrm{u} \in H^1 \times H^{1/2}$, $\mathrm{p} \in L^2(\Omega_1)^{3\times 3}_{dev} \times L^2(\Omega_2)^{3\times 3}_{dev}$, $\beta \in H^1(\Gamma)$. For short we will use the notation $\langle \cdot, \cdot \rangle$ to indicate any of the products

$$\langle \cdot, \cdot \rangle_{L^2(\Omega_1)^3 \times L^2(\Omega_2)^3} \mathrm{or} \langle \cdot, \cdot \rangle_{L^2(\Omega_1)^{3\times 3} \times L^2(\Omega_2)^{3\times 3}}.$$

Theorem 5. **V** *and* **W** *are Hilbert space under the inner product*

$$\left\langle (u, p, \beta)_V, (\hat{u}, \hat{p}, \hat{\beta})_V \right\rangle_V := \langle u, \hat{u} \rangle_{H^1} + \langle \gamma_0 u, \gamma_0 \hat{u} \rangle_{H^{\frac{1}{2}}} + \langle p, \hat{p} \rangle + \left\langle \beta, \hat{\beta} \right\rangle_{H^1(\Gamma)},$$

$$\left\langle (u, p, \beta)_W, (\hat{u}, \hat{p}, \hat{\beta})_W \right\rangle_W := \langle u, \hat{u} \rangle + \langle p, \hat{p} \rangle + \left\langle \beta, \hat{\beta} \right\rangle_{L^2(\Gamma).}$$

Besides that, **V**⋐**W**.

Theorem 6. *If* u, p, β *satisfy* (2)–(9),

$$\mathrm{q} = (q^1, q^2) \in \left(\partial R(\dot{p}^1), \partial R(\dot{p}^2) \right),$$

$$\varrho \in \partial I_-(\mathrm{u}_{|\Gamma}(^{-1}_{\ 1}) \cdot N_2),$$

$$\alpha \in \partial I_-(\dot{\beta}),$$

$$\varsigma \in \partial I_{[0,1]}(\beta),$$

then for all $(\hat{\mathrm{u}}, \hat{\mathrm{p}}, \hat{\beta})_V \in \mathbf{V}$,

$$\langle \partial_e W(\mathrm{e(u)}, \mathrm{p}), \mathrm{e}(\hat{\mathrm{u}}) \rangle - \int_\Gamma \sigma^i N^i \cdot \hat{u}^i ds = \mathrm{f}, \hat{\mathrm{u}} + \left\langle \mathrm{g}, \hat{\mathrm{u}}_{|\partial\Omega\backslash\Gamma} \right\rangle_{\partial\Omega\backslash\Gamma} \tag{11}$$

$$\langle \mathrm{q}, \hat{\mathrm{p}} \rangle + \left\langle \partial_p W(\mathrm{e(u)}, \mathrm{p}), \hat{\mathrm{p}} \right\rangle = 0 \tag{12}$$

$$\int_\Gamma \sigma^i N_i \cdot \hat{\mathrm{u}}^i ds + k\langle \beta \mathrm{u}_{|\Gamma}(^{-1}_{\ 1}), \hat{\mathrm{u}}_{|\Gamma}(^{-1}_{\ 1}) \rangle_\Gamma + \int_\Gamma \varrho \hat{\mathrm{u}}_{|\Gamma}(^{-1}_{\ 1}) \cdot N_2 ds = 0 \tag{13}$$

$$\left\langle c_s \dot{\beta} + \alpha + \varsigma, \hat{\beta} \right\rangle_\Gamma + k_s \langle \nabla\beta, \nabla\hat{\beta} \rangle_\Gamma + \left\langle \frac{k}{2} \left\| \mathrm{u}_{|\Gamma}(^{-1}_{\ 1}) \right\|^2, \hat{\beta} \right\rangle_\Gamma = \langle w_s, \hat{\beta} \rangle_\Gamma \tag{14}$$

Proof. By Green's formula,

$$\left\langle \sigma^i, e(\hat{u}_i) \right\rangle + \left\langle div\sigma^i, \hat{u}^i \right\rangle = \int_{\partial\Omega_i} \sigma^i N_i \cdot \hat{u}^i ds. \tag{15}$$

Substituting (2) into (15), where $\sigma^i = \partial_e W(e(u^i), p^i)$, we obtain

$$\int_{\Omega_i} f^i \cdot \hat{u}^i = -\left\langle div\sigma^i, \hat{u}^i \right\rangle = \left\langle \partial_e W(e(u^i), p^i), e(\hat{u}^i) \right\rangle - \int_{\partial\Omega_i} \sigma^i N_i \cdot \hat{u}^i ds. \tag{16}$$

From equation (5),

$$\int_{\partial\Omega_i} \sigma^i N_i \cdot \hat{u}^i ds = \int_\Gamma \sigma^i N_i \cdot \hat{u}^i ds + \int_{\partial\Omega_i\backslash\Gamma} \sigma^i N_i \cdot \hat{u}^i ds$$
$$= \int_\Gamma \sigma^i N_i \cdot \hat{u}^i ds + \int_{\partial\Omega_i\backslash(\Gamma\cup\Gamma_i)} g^i \cdot \hat{u}^i, \tag{17}$$

and from the substitution of (17) in (16),

$$\langle \partial_e W(\mathrm{e(u)}, \mathrm{p}), \mathrm{e}(\hat{\mathrm{u}}) \rangle - \int_\Gamma \sigma^i N_i \cdot \hat{u}^i ds = \langle \mathrm{f}, \hat{\mathrm{u}} \rangle + \langle \mathrm{g}, \hat{\mathrm{u}} \rangle_{\partial\Omega\backslash\Gamma}.$$

Equation (12) is immediate by (3). Multiplying the first equation of (7) by $\hat{u}^1$, the second by $\hat{u}^2$, and integrating over Γ,

$$\begin{aligned}&\int_\Gamma (\sigma^1 N_1 \cdot \hat{u}^1 + \sigma^2 N_2 \cdot \hat{u}^2) + k\beta(u^2 - u^1)\cdot(\hat{u}^2 - \hat{u}^1) \\ &\qquad + \varrho(\hat{u}^2 - \hat{u}^1)\cdot N_2 ds \\ &= \int_\Gamma \sigma^i N_i \cdot \hat{u}^i ds + k\beta \mathrm{u}_{|\Gamma}\binom{-1}{1}, \hat{\mathrm{u}}_{|\Gamma}\binom{-1}{1})_\Gamma \\ &\qquad + \int_\Gamma \varrho \hat{\mathrm{u}}_{|\Gamma}\binom{-1}{1}\cdot N_2 ds \\ &= 0,\end{aligned}$$

where $\varrho \in \partial I_-(\mathrm{u}_{|\Gamma}\binom{-1}{1} \cdot N_2), \hat{\mathrm{u}} = (\hat{u}^1, \hat{u}^2) \in H^1$.

If $\beta \in H^1(\Gamma), \mathrm{u} \in H^1, c_s\dot{\beta} + \alpha \in \partial\left(\frac{c_s}{2}\dot{\beta}^2 + I_-(\dot{\beta})\right)$ and $\varsigma \in \partial I_{[0,1]}(\beta)$ satisfy the relation (6), then for all $\hat{\beta} \in H^1_\Gamma$

$$\begin{aligned}&\left\langle c_s\dot{\beta} + \alpha, \hat{\beta}\right\rangle_\Gamma - k_s\langle \Delta\beta, \hat{\beta}\rangle_\Gamma + \langle \varsigma, \hat{\beta}\rangle_\Gamma \\ &\qquad = \langle w_s, \hat{\beta}\rangle_\Gamma - \frac{1}{2}k\left\langle \|u^2 - u^1\|^2, \hat{\beta}\right\rangle_\Gamma,\end{aligned}$$

and by condition (8),

$$\begin{aligned}&\left\langle c_s\dot{\beta} + \alpha, \hat{\beta}\right\rangle_\Gamma + k_s\langle \nabla\beta, \nabla\hat{\beta}\rangle_\Gamma + \langle \varsigma, \hat{\beta}\rangle_\Gamma \\ &\qquad = \langle w_s, \hat{\beta}\rangle_\Gamma - \frac{k}{2}\left\langle \|\mathrm{u}_{|\Gamma}\binom{-1}{1}\|^2, \hat{\beta}\right\rangle_\Gamma.\end{aligned}$$

We will write the equations of the theorem 6 so that the adhesive contact model can be represented as a doubly non-linear problem. The sum of the terms (11)–(14) will be grouped taking into account the pairings $(\mathbf{V}, \mathbf{V}^*)$, and $(\mathbf{W}, \mathbf{W}^*)$:

$$\begin{aligned}&\left\{\langle 0, \hat{\mathrm{u}}\rangle + \langle \mathrm{q}, \hat{\mathrm{p}}\rangle + \left\langle c_s\dot{\beta} + \alpha, \hat{\beta}\right\rangle_\Gamma\right\} \\ &+ \left\{\begin{array}{l}\langle 0, \hat{\mathrm{u}}\rangle + \langle \partial_e W(\mathrm{e(u)}, \mathrm{p}), \mathrm{e}(\hat{\mathrm{u}})\rangle \\ + \left\langle 0, \hat{\mathrm{u}}_{|\partial\Omega\backslash\Gamma}\right\rangle_{\partial\Omega\backslash\Gamma} + k\langle \beta \mathrm{u}_{|\Gamma}\binom{-1}{1}, \hat{\mathrm{u}}_{|\Gamma}\binom{-1}{1}\rangle_\Gamma \\ + \int_\Gamma \varrho \hat{\mathrm{u}}_{|\Gamma}\binom{-1}{1}\cdot N_2 ds + \langle \partial_p W(\mathrm{e(u)}, \mathrm{p}), \hat{\mathrm{p}}\rangle \\ + \left\langle \varsigma + \|\mathrm{u}_{|\Gamma}\binom{-1}{1}\|^2, \hat{\beta}\right\rangle_\Gamma + k_s\left\langle \nabla\beta, \nabla\hat{\beta}\right\rangle_\Gamma\end{array}\right\} \\ &= \left\{\begin{array}{c}\langle \mathrm{f}, \hat{\mathrm{u}}\rangle + \langle 0, \mathrm{e}(\hat{\mathrm{u}})\rangle \\ + \left\langle \mathrm{g}, \hat{\mathrm{u}}_{|\partial\Omega\backslash\Gamma}\right\rangle_{\partial\Omega\backslash\Gamma} + \langle 0, \hat{\mathrm{u}}_{|\Gamma}\rangle_\Gamma \\ + \langle 0, \hat{\mathrm{p}}\rangle + \left\langle \omega_s, \hat{\beta}\right\rangle_\Gamma + \left\langle 0, \nabla\hat{\beta}\right\rangle_\Gamma\end{array}\right\},\end{aligned}$$

where $\mathrm{f} = (f^1, f^2), \mathrm{g} = (g^1, g^2)$. The first summation corresponds to the product $\left\langle (0, \mathrm{q}, c_s\dot{\beta} + \alpha)_W, (\hat{\mathrm{u}}, \hat{\mathrm{p}}, \hat{\beta})_W\right\rangle_W$, where

$$(0, \mathrm{q}, c_s\dot{\beta} + \alpha)_W \in \partial_W\left\{R(\dot{\mathrm{p}}) + \frac{c_s}{2}\left(\dot{\beta}\right)^2 + I_-(\dot{\beta})\right\}, \tag{18}$$

while the second adding corresponds to the product $\langle v, (\hat{\mathrm{u}}, \hat{\mathrm{p}}, \hat{\beta})_V\rangle_V$, where

$$\begin{aligned}v \in \partial_V W(\mathrm{e}, \mathrm{p}) + \frac{k_s}{2}\|\nabla\beta\|^2 + \frac{k}{2}\beta\|\mathrm{u}_{|\Gamma}\binom{-1}{1}\|^2 \\ + I_{[0,1]}(\beta) + I_-(\mathrm{u}_{|\Gamma}\binom{-1}{1}\cdot N_2).\end{aligned} \tag{19}$$

The last term is the product between F and $(\hat{\mathrm{u}}, \hat{\mathrm{p}}, \hat{\beta})_V$, where

$$F := \begin{pmatrix}\mathrm{f} & 0 & \mathrm{g} & 0 \\ & 0 & & \\ & \omega_s & 0 & \end{pmatrix}.$$

Definition 7. *We define by* $\varphi : \mathbf{W} \to]-\infty, \infty]$, *and* $\psi : \mathbf{V} \to]-\infty, \infty]$ *the functional*

$$\varphi(\mathrm{u}, \mathrm{p}, \beta)_W = R(\mathrm{p}) + \frac{c_s}{2}(\beta)^2 + I_-(\beta),$$

$$\begin{aligned}\psi(\mathrm{u}, \mathrm{p}, \beta)_V = W(\mathrm{e}, \mathrm{p}) + \frac{k_s}{2}\|\nabla\beta\|^2 + \frac{k}{2}\beta\|\mathrm{u}_{|\Gamma}\binom{-1}{1}\|^2 \\ + I_{[0,1]}(\beta) + I_-(\mathrm{u}_{|\Gamma}\binom{-1}{1}\cdot N_2),\end{aligned}$$

and their respective subdifferentials $\partial_W\varphi : \mathbf{W} \rightrightarrows \mathbf{W}^*$, *and* $\partial_V\psi : \mathbf{V} \rightrightarrows \mathbf{V}^*$ *by*

$$\partial_W\varphi(\mathrm{u}, \mathrm{p}, \beta)_W = \begin{pmatrix}0 \\ \partial_p R(\mathrm{p}) \\ c_s\beta + \partial_\beta I_-(\beta)\end{pmatrix},$$

$$\partial_V\psi(\mathrm{u}, \mathrm{p}, \beta)_V = \begin{pmatrix}0 \quad \partial_e W(\mathrm{e(u)}, \mathrm{p}) \quad 0 \quad \partial_{u_{|\Gamma}}\left\{\frac{k}{2}\beta\|\mathrm{u}_{|\Gamma}\binom{-1}{1}\|^2 + I_-(\mathrm{u}_{|\Gamma}\binom{-1}{1}\cdot N_2)\right\} \\ \partial_p W(\mathrm{e(u)}, \mathrm{p}) \\ \partial_\beta I_{[0,1]}(\beta) + \frac{k}{2}\|\mathrm{u}_{|\Gamma}\binom{-1}{1}\|^2 \qquad k_s\nabla\beta\end{pmatrix}$$

Definition 8. *For* $(\mathrm{f}, 0) \in (H^1)^*, (\mathrm{g}, 0) \in (H^{1/2})^*$, $\omega_s \in (L^2(\Gamma))^*$, *we define the mapping* $F \in \mathbf{V}^*$ *by*

$$\langle F, (\mathrm{u}, \mathrm{p}, \beta)_V\rangle_V = \langle \mathrm{f}, \mathrm{u}\rangle + \left\langle \mathrm{g}, \mathrm{u}_{|\partial\Omega\backslash\Gamma}\right\rangle_{\partial\Omega\backslash\Gamma} + \langle \omega_s, \beta\rangle_\Gamma$$

for each $(\mathrm{u}, \mathrm{p}, \beta)_V \in \mathbf{V}$.

Problem 9. The weak problem of contact with adhesion and elastoplastic deformation is defined by:

Find u, p, β such that

$$\partial_W \varphi(\dot{\mathrm{u}}, \dot{\mathrm{p}}, \dot{\beta})_W + \partial_V \psi(\mathrm{u}, \mathrm{p}, \beta)_V \ni F \tag{20}$$

Definition 10. *A solution of the Problem 9 is a triplet* $(\mathrm{u}, \mathrm{p}, \beta) \in H^1(0, T; \mathbf{V})$ *satisfying* (20).

Existence of weak solutions

Theorem 11. $\partial\psi$ *is strongly monotone*

Before making this proof, we will prove that

Theorem 12. *The mapping* $(\mathrm{u}, \mathrm{p}) \mapsto \langle \partial_e W(\mathrm{e}(\mathrm{u}), \mathrm{p}), \mathrm{e}(\mathrm{u})\rangle + \langle \partial_p W(\mathrm{e}(\mathrm{u}), \mathrm{p}), \mathrm{p}\rangle$ *is strongly monotone.*

Proof. Since $\partial_e W(\mathrm{e}(\mathrm{u}), \mathrm{p})$ and $\partial_p W(\mathrm{e}(\mathrm{u}), \mathrm{p})$ are linear for all $(\mathrm{u}, \mathrm{p}) \in \mathbf{V}$, we will prove that there exists $C>0$ such that

$$\langle \partial_e W(\mathrm{e}(\mathrm{u}), \mathrm{p}), \mathrm{e}(\mathrm{u})\rangle + \langle \partial_p W(\mathrm{e}(\mathrm{u}), \mathrm{p}), \mathrm{p}\rangle \geqslant C\|(\mathrm{u}, \mathrm{p})\|_V^2.$$

For short we will symbolize $\mathrm{e}(\mathrm{u}) = \mathrm{e}$, and $e(u^i) = e^i$. From equation (4), $\partial_e W(e(u^i), p^i) = C(e(u^i) - p^i)$ and $\partial_p W(e(u^i), p^i) = -C(e(u^i) - p^i) + h_i p^i$. Using the Einstein summation convention,

$$\begin{aligned}
&\langle \partial_e W(\mathrm{e}(\mathrm{u}), \mathrm{p}), \mathrm{e}(\mathrm{u})\rangle + \langle \partial_p W(\mathrm{e}(\mathrm{u}), \mathrm{p}), \mathrm{p}\rangle \\
&= \langle C(e(u^i) - p^i), e(u^i) - p^i\rangle + h_i\|p^i\|^2 \\
&\geqslant 2\mu_i\|e^i - p^i\|^2 + h_i\|p^i\|^2 \\
&\geqslant min\left\{\mu, \frac{h}{2}\right\}\left[\|e^i - p^i\|^2 + \|p^i\|^2\right] + \frac{h}{2}\|p^i\|^2 \\
&\geqslant \mathrm{c}\left\{[\|\mathrm{e} - \mathrm{p}\| + \|\mathrm{p}\|]^2 + \|\mathrm{p}\|^2\right\} \geqslant \mathrm{c}\left\{\|\mathrm{e}\|^2 + \|\mathrm{p}\|^2\right\},
\end{aligned}$$

and by the Korn's inequality ,

$$\begin{aligned}
\mathrm{c}\left\{\|\mathrm{e}\|^2 + \|\mathrm{p}\|^2\right\} &\geqslant c\left\{ C_K\|\mathrm{u}\|_{\mathrm{H}^1}^2 + \|\mathrm{p}\|^2\right\} \\
&\geqslant C\left\{\|\mathrm{u}\|_{H^1 \times H^{1/2}}^2 + \|\mathrm{p}\|^2\right\}.
\end{aligned}$$

By the linearity of e, $\partial_e W$ and $\partial_p W$,

$$\begin{aligned}
&\langle \partial_e W(\mathrm{e}(\hat{\mathrm{u}} - \mathrm{u}), \hat{\mathrm{p}} - \mathrm{p}), \mathrm{e}(\hat{\mathrm{u}} - \mathrm{u})\rangle \\
&\quad + \langle \partial_p W(\mathrm{e}(\hat{\mathrm{u}} - \mathrm{u}), \hat{\mathrm{p}} - \mathrm{p}), \hat{\mathrm{p}} - \mathrm{p}\rangle \\
&\quad \geqslant C\left\{\|\hat{u} - \mathrm{u}\|_{H^1 \times H^{1/2}}^2 + \|\hat{\mathrm{p}} - \mathrm{p}\|^2\right\}.
\end{aligned}$$

Proof of Theorem 11. Because $\frac{k}{2}\beta\left\|\mathrm{u}_{|_\Gamma}\binom{-1}{1}\right\|^2, I_{[0,1]}(\beta)$, and $I_-(\mathrm{u}_{|_\Gamma}\binom{-1}{1} \cdot N_2)$ are convex, proper and l.s.c,

$$\partial_V\left\{\frac{k}{2}\beta\left\|\mathrm{u}_{|_\Gamma}\binom{-1}{1}\right\|^2 + I_{[0,1]}(\beta) + I_-(\mathrm{u}_{|_\Gamma}\binom{-1}{1} \cdot N_2)\right\} \tag{21}$$

is monotone. By Korn's inequality over Γ,

$$\langle k_s\nabla(\beta_1 - \beta_2), \nabla(\beta_1 - \beta_2)\rangle_\Gamma \geqslant C\|\beta_1 - \beta_2\|_{H^1(\Gamma)}^2, \tag{22}$$

therefore, by the theorem 12, the equations (21), (22), if $\xi_1 \in \partial\psi(\mathrm{u}, \mathrm{p}, \beta)_{1_V}$, $\xi_2 \in \partial\psi(\mathrm{u}, \mathrm{p}, \beta)_{2_V}$, then

$$\begin{aligned}
&\langle \xi_1 - \xi_2, (\mathrm{u}, \mathrm{p}, \beta)_{1_V} - (\mathrm{u}, \mathrm{p}, \beta)_{2_V}\rangle \\
&\geqslant C\left\{\|\mathrm{u}_1 - \mathrm{u}_2\|_{H^1 \times H^{\frac{1}{2}}}^2 + \|\mathrm{p}_1 - \mathrm{p}_2\|^2\right\} + C\|\beta_1 - \beta_2\|_{H^1(\Gamma)}^2 \\
&= C\left\|(\mathrm{u}, \mathrm{p}, \beta)_{1_V} - (\mathrm{u}, \mathrm{p}, \beta)_{2_V}\right\|_V^2, \; C>0,
\end{aligned}$$

and $\partial\psi$ is strongly monotone.

We want to guarantee the existence of solutions through the Theorem 4, but not directly since $\partial_W\varphi$ is not bounded. So, we will prove the existence of solutions of the following differential inclusion:

$$\begin{aligned}
&\partial_W \varphi_n(\dot{\mathrm{u}}, \dot{\mathrm{p}}, \dot{\beta})_W + \partial_V \psi(\mathrm{u}, \mathrm{p}, \beta)_V \ni F, \\
&(\mathrm{u}, \mathrm{p}, \beta)_V(0) = (\mathrm{u}_0, \mathrm{p}_0, \beta_0)_V,
\end{aligned} \tag{23}$$

with $\partial_W\varphi_n$ bounded, where

$$\varphi_n(\mathrm{u}, \mathrm{p}, \beta)_W = R(\mathrm{p}) + \frac{c_s}{2}\|\beta\|_\Gamma^2 + I_-^n(\beta),$$

$$I_-^n(\beta) = \begin{cases} 0 & if\ \beta(\Gamma) \subset]-\infty, 0] \\ n\|\beta\|^+{}_\Gamma & \text{otherwise} \end{cases},$$

$$\psi_n(\mathrm{u}, \mathrm{p}, \beta)_V \equiv \psi(\mathrm{u}, \mathrm{p}, \beta)_V.$$

Later it will be proved that this solution also solves problem 9. In the rest of the document we will symbolize

$$z(0) = z_0 = \begin{pmatrix} \mathrm{u}_0 & \mathrm{e}(\mathrm{u}_0) & \mathrm{u}_{0|_{\partial\Omega\setminus\Gamma}} & \mathrm{u}_{0|_\Gamma} \\ & & \mathrm{p}_0 & \\ & \beta_0 & \nabla\beta_0 & \end{pmatrix} \in V,$$

$$v_0 \in \partial_V\psi(u_0, p_0, \beta_0)_V,$$

and

$$F(0) = \begin{pmatrix} \mathrm{f}(0) & 0 & \mathrm{g}(0) & 0 \\ & 0 & & \\ & \omega_s(0) & 0 & \end{pmatrix}.$$

Theorem 13. *If* $F(0) - v_0 \in \partial_W\{R(\mathrm{p}) + \partial I_-^n(\beta)\}_{|(\mathrm{p},\beta)=(0,0)}$, *then for each* $k \geqslant n \in \mathbb{N}$,

$$F(0) - v_0 \in D(\varphi_k^*).$$

Proof. Since $\partial_u\varphi = 0$, $\mathrm{f}(0) = 0$ must be assumed. Consider the representation

$$v_{0_W} = \begin{pmatrix} v_{0_1} \\ v_{0_2} \\ v_{0_3} \end{pmatrix} \text{ where } \begin{cases} v_{0_2} = \partial_p W(e(\mathrm{u}_0), \mathrm{p}_0), \\ v_{0_3} \in \partial_\beta I_{[0,1]}(\beta_0) + \frac{k}{2}\left\|\mathrm{u}_{0|_\Gamma}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\right\|^2 . \end{cases}$$

Since $F(0)_2 - v_{0_2} \in \partial_p R(0)$ and $F(0)_3 - v_{0_3} \in \partial I_-^n(0)$, then for all $(\mathrm{u},\mathrm{p},\beta)_W \in \mathbf{W}$,

$$\langle F(0)_2 - v_{0_2}, \mathrm{p}\rangle \leqslant R(\mathrm{p}) - R(0),$$

$$\langle F(0)_3 - v_{0_3}, \beta\rangle_\Gamma \leqslant I_-^n(\beta) - I_-^n(0) \leqslant I_-^k(\beta) - I_-^k(0),$$

for each $k \geqslant n$. So

$$\begin{aligned} \langle F(0) - v_0, (\mathrm{u},\mathrm{p},\beta)_W\rangle_W &- \varphi_k(\mathrm{u},\mathrm{p},\beta)_W \\ &= \langle F(0)_2 - v_{0_2}, \mathrm{p}\rangle + \langle F(0)_3 - v_{0_3}, \beta\rangle_\Gamma \\ &\qquad - R(\mathrm{p}) - I_-^k(\beta) \\ &\leqslant R(0) + I_-^k(0) < \infty, \end{aligned}$$

and $F(0) - v_0 \in D(\varphi_k^*)$ for all $k \geqslant n \in \mathbb{N}$.

Theorem 14. *Given* $z_0 \in \mathbf{V}, v_0 \in \partial\psi(z_0), F \in L^1(0,T;\mathbf{W}^*) \cap H^1(0,T;\mathbf{V}^*)$ *with the hypotheses of the Theorem 13, then there is a weak solution to the problem 9.*

Proof. It is not difficult to verify that $\varphi_\mathrm{n}(\mathrm{u},\mathrm{p},\beta)_\mathrm{W} = \mathrm{R}(\mathrm{p}) + \frac{c_s}{2}\|\beta\|_\Gamma^2 + \mathrm{I}_-^\mathrm{n}(\beta)$ and $\psi_\mathrm{n}(\mathrm{u},\mathrm{p},\beta)_\mathrm{V} = \psi(\mathrm{u},\mathrm{p},\beta)_\mathrm{V}$ are proppers, l.s.c., and convex functions. Further, $\partial\varphi_n$ is monotone (see e.g. Showalter,[27] p. 158) and bounded. By Theorem 13, $F(0) - v_0 \in D(\varphi_n^*)$for all $n \geqslant N \in \mathbb{N}$, and by Theorem 11 $\partial\psi$ is strongly monotone. By Theorem 4, for each $n \geqslant N$ there exist $z_n \in H^1(0,T;\mathbf{V}), w_n \in L^\infty(0,T;\mathbf{W}^*), v_n \in L^1(0,T;\mathbf{W}^*) \cap L^\infty(0,T;\mathbf{V}^*)$ that satisfy differential inclusion (23), this is,

$$w_n(t) + v_n(t) = F(t),$$

$$w_n(t) \in \partial\varphi_n\left(\dot{z}_n(t)\right),$$

$$v_n(t) \in \partial\psi_n(z_n(t)) \textit{for a.e. } t \in]0,T[,$$

$$z_n(0) = z_0.$$

If for some $m \in \mathbb{N}$ we prove that $\underset{m}{\dot{z}} \in D(\varphi)$, then we will have proved that z_m is a weak solution to the problem 9.

We must assume $\{n \in \mathbb{N} : \beta_n = 0\} = \emptyset$ (otherwise $\dot{\beta}_n = 0$ and a weak solution to the problem is obtained). Therefore, $n \in \mathbb{N}$ implies $0 \neq \beta_n \leqslant 1$, and $\varsigma_n \geqslant 0$ for each $\varsigma_n \in \partial I_{[0,1]}(\beta_n)$.

If $\#\left\{n \in \mathbb{N} : \dot{\beta}_n^+ > 0\right\} = \infty$, then there is a sub-sequence $\dot{\beta}_{n_i}$ of $\dot{\beta}_n$, and a sub-sequence $\varsigma_{n_i} \in \partial I_{[0,1]}(\beta_{n_i})$, such that $\partial I_-^{n_i}(\dot{\beta}_{n_i}) = \frac{n_i}{\left\|\dot{\beta}_{n_i}^+\right\|_\Gamma}\dot{\beta}_{n_i}^+$ and $R(\varsigma_{n_i}) \in [0, +\infty[$. It is clear that $\dot{\beta}_{n_i}^+ \in L^2(\Gamma)$, since $z_{n_i} \in H^1(0,T;\mathbf{V})$. Multiplying by $\dot{\beta}_{n_i}^+$ the component β of the inclusion (23),

$$\begin{aligned} \left\langle \omega_s, \dot{\beta}_{n_i}^+ \right\rangle_\Gamma &\geqslant \left\langle c_s\dot{\beta}_{n_i} + \frac{n_i}{\left\|\dot{\beta}_{n_i}^+\right\|_\Gamma}\dot{\beta}_{n_i}^+, \dot{\beta}_{n_i}^+ \right\rangle_\Gamma \\ &\quad + \left\langle \varsigma_{n_i}, \dot{\beta}_{n_i}^+ \right\rangle_\Gamma + \frac{k}{2}\left\langle \left\|\mathrm{u}_{|_\Gamma}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\right\|^2, \dot{\beta}_{n_i}^+ \right\rangle_\Gamma \\ &\geqslant \frac{n_i}{\left\|\dot{\beta}_{n_i}^+\right\|_\Gamma}\left\langle \dot{\beta}_{n_i}^+, \dot{\beta}_{n_i}^+ \right\rangle_\Gamma, \end{aligned}$$

so $\|\omega_s\|_\Gamma \geqslant \left\langle \omega_s, \dot{\beta}_{n_i}^+ / \left\|\dot{\beta}_{n_i}^+\right\|_\Gamma \right\rangle \underset{n_i\to\infty}{\to} \infty$, which contradicts that $F \in H^1(0,T;\mathbf{V}^*)$. For this reason it follows that

$$\#\left\{n \in N : \dot{\beta}_n^+ > 0\right\} < \infty,$$

and there exists $M \in \mathbb{N}$ such that $\dot{\beta}_m \leqslant 0$ for all $m \geqslant M$. This proves that $\underset{m}{\dot{z}} \in D(\varphi)$, and since $\partial I_-^n(\dot{\beta}_m) \subset \partial I_-(\dot{\beta}_m)$ for all $\dot{\beta}_m \leqslant 0$, it follows that z_m is a weak solution to the problem 9.

Summary and conclusions

- Differential inclusions that formulate a weak uni-directional adhesive unilateral contact problem, with elastoplastic deformation and hardening, were compressed as an abstract doubly non-linear problem, with unbounded multivalued operators.
- The weak formulation allowed the derivative for displacement and adhesion field to be reduced from order 2 to order 1.
- The model covers both, the rate-independent case as well as the mixed case (rate-independent for elastoplasticity and rate-dependent for adhesion).
- The geometric requirements on the boundaries of deformable solids demand that they be of class C and that they also satisfy the Korn inequality both for the displacements and for the bonding

field; For this, null displacement is required in a part of the boundary of each solid.

- The displacement variable was extended to the contact boundary to interact as an inner product with the bonding field.
- The paper proves the existence of weak solutions of the model without using the energy solutions approach in similar papers. To obtain the proof of existence we construct a succession of doubly nonlinear problems that approximate the model under study. It was proved that each problem in the sequence has a solution, and that one of these models shares a solution with the original model.

Declaration of conflicting interests

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This Research had the support of the Universidad del Atlántico and its Doctoral Training Program for professors - Resolution 1735 of October 13, 2016. Open access license granted by SAGE Publishing.

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